AUTHORIZED

TEXT BOOK SERIES

TODHUNTER'S EUCLID

FOR SCHOOLS & COLLEGES

BOOKS, I & II.

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TORONTO

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Canadian Series of School Books.

THE ELEMENTS OF

EUCLID

FOR THE USE OF SCHOOLS AND COLLEGES;

COMPRISING THE FIRST TWO BOOKS AND PORTIONS OF THE ELEVENTH AND TWELFTH BOOKS;

WITH NOTES AND EXERCISES.

I. TODHUNTER, M.A., F.R.S

NEW EDITION.

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EUCLID'S ELEMENTS.

BOOK I.

DEFINITIONS.

1. A POINT is that which has no parts, or which has no magnitude.

2. A line is length without breadth.

3. The extremities of a line are points.

4. A straight line is that which lies evenly between its extreme points.

5. A superficies is that which has only length and breadth.

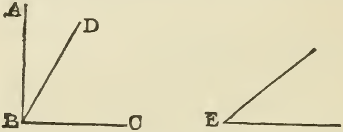
6. The extremities of a superficies are lines.

7. A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

8. A plane angle is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

9. A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

Note. When several angles are at one point *B*, any one of them is expressed by three letters, of which the letter which is at the vertex of the angle, that is, at the point at which the straight lines that contain the angle meet one another, is put between the other two letters, and one of these two letters is somewhere on one of those straight lines, and the other letter on the other straight line. Thus, the angle which is contained by the

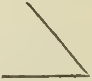


straight lines *AB*, *CB* is named the angle *ABC*, or *CBA*; the angle which is contained by the straight lines *AB*, *DB* is named the angle *ABD*, or *DBA*; and the angle which is contained by the straight lines *DB*, *CB* is named the angle *DBC*, or *CBD*; but if there be only one angle at a point, it may be expressed by a letter placed at that point; as the angle at *E*.

10. When a straight line standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.

11. An obtuse angle is that which is greater than a right angle.

12. An acute angle is that which is less than a right angle.



13. A term or boundary is the extremity of any thing.

14. A figure is that which is enclosed by one or more boundaries.

15. A circle is a plane figure contained by one line, which is called the circumference, and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another:

16. And this point is called the centre of the circle.

17. A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

[A radius of a circle is a straight line drawn from the centre to the circumference.]

18. A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

19. A segment of a circle is the figure contained by a straight line and the circumference which it cuts off.

20. Rectilineal figures are those which are contained by straight lines:

21. Trilateral figures, or triangles, by three straight lines:

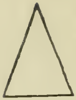
22. Quadrilateral figures by four straight lines:

23. Multilateral figures, or polygons, by more than four straight lines.

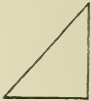
24. Of three-sided figures,

An equilateral triangle is that which has three equal sides:

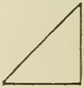
25. An isosceles triangle is that which has two sides equal:



26. A scalene triangle is that which has three unequal sides:



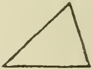
27. A right-angled triangle is that which has a right angle:



[The side opposite to the right angle in a right-angled triangle is frequently called the hypotenuse.]

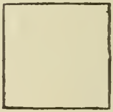
28. An obtuse-angled triangle is that which has an obtuse angle:

29. An acute-angled triangle is that which has three acute angles.



Of four-sided figures,

30. A square is that which has all its sides equal, and all its angles right angles:

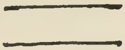


31. An oblong is that which has all its angles right angles, but not all its sides equal:

32. A rhombus is that which has all its sides equal, but its angles are not right angles:

33. A rhomboid is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles:

34. All other four-sided figures besides these are called trapeziums.



35. Parallel straight lines are such as are in the same plane, and which being produced ever so far both ways do not meet.

[*Note.* The terms *oblong* and *rhomboid* are not often used. Practically the following definitions are used. Any four-sided figure is called a *quadrilateral.* A line joining two opposite angles of a quadrilateral is called a *diagonal.* A quadrilateral which has its opposite sides parallel is called a *parallelogram.* The words *square* and *rhombus* are used in the sense defined by Euclid; and the word *rectangle* is used instead of the word *oblong.*

Some writers propose to restrict the word *trapezium* to a quadrilateral which has two of its sides parallel; and it would certainly be convenient if this restriction were universally adopted.]

POSTULATES.

Let it be granted,

1. That a straight line may be drawn from any one point to any other point:

2. That a terminated straight line may be produced to any length in a straight line:

3. And that a circle may be described from any centre, at any distance from that centre.

AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal.

3. If equals be taken from equals the remainders are equal.

4. If equals be added to unequals the wholes are unequal.

5. If equals be taken from unequals the remainders are unequal.

6. Things which are double of the same thing are equal to one another.

7. Things which are halves of the same thing are equal to one another.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than its part.

10. Two straight lines cannot enclose a space.

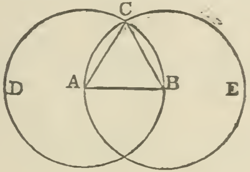
11. All right angles are equal to one another.

12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

PROPOSITION 1. *PROBLEM.*

To describe an equilateral triangle on a given finite straight line.

Let AB be the given straight line: it is required to describe an equilateral triangle on AB.



From the centre A, at the distance AB, describe the circle BCD. [*Postulate* 3.

From the centre B, at the distance BA, describe the circle ACE. [*Postulate* 3.

From the point C, at which the circles cut one another, draw the straight lines CA and CB to the points A and B. [*Post.* 1.

ABC shall be an equilateral triangle.

Because the point A is the centre of the circle BCD, AC is equal to AB. [*Definition* 15.

And because the point B is the centre of the circle ACE, BC is equal to BA. [*Definition* 15.

But it has been shewn that CA is equal to AB;

therefore CA and CB are each of them equal to AB.

But things which are equal to the same thing are equal to one another. [*Axiom* 1.

Therefore CA is equal to CB.

Therefore CA, AB, BC are equal to one another.

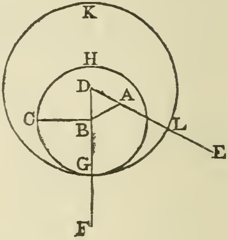
Wherefore *the triangle ABC is equilateral,* [*Def.* 24.

and it is described on the given straight line AB. Q.E.F.

PROPOSITION 2. *PROBLEM.*

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line: it is required to draw from the point A a straight line equal to BC.



From the point A to B draw the straight line AB; [*Post.* 1.
and on it describe the equilateral triangle DAB, [I. 1.
and produce the straight lines DA, DB to E and F. [*Post.* 2.
From the centre B, at the distance BC, describe the circle CGH, meeting DF at G. [*Post.* 3.
From the centre D, at the distance DG, describe the circle GKL, meeting DE at L. [*Post.* 3.
AL shall be equal to BC.

Because the point B is the centre of the circle CGH, BC is equal to BG. [*Definition* 15.
And because the point D is the centre of the circle GKL, DL is equal to DG; [*Definition* 15.
and DA, DB parts of them are equal; [*Definition* 24.
therefore the remainder AL is equal to the remainder BG. [*Axiom* 3.
But it has been shewn that BC is equal to BG;
therefore AL and BC are each of them equal to BG.
But things which are equal to the same thing are equal to one another. [*Axiom* 1.
Therefore AL is equal to BC.

Wherefore *from the given point A a straight line AL has been drawn equal to the given straight line BC.* Q.E.F.

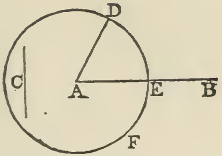
PROPOSITION 3. *PROBLEM.*

From the greater of two given straight lines to cut off a part equal to the less

Let AB and C be the two given straight lines, of which

AB is the greater: it is required to cut off from AB, the greater, a part equal to C the less.

From the point A draw the straight line AD equal to C; [I. 2.



and from the centre A, at the distance AD, describe the circle DEF meeting AB at E. [*Postulate* 3.

AE shall be equal to C.

Because the point A is the centre of the circle DEF, AE is equal to AD. [*Definition* 15.

But C is equal to AD. [*Construction*.

Therefore AE and C are each of them equal to AD.

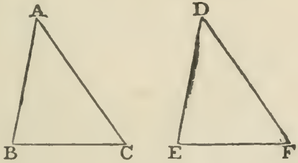
Therefore AE is equal to C. [*Axiom* 1.

Wherefore *from AB the greater of two given straight lines a part AE has been cut off equal to C the less.* Q.E.F.

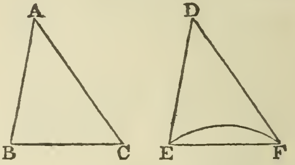
PROPOSITION 4. *THEOREM.*

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases or third sides equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

Let ABC, DEF be two triangles which have the two sides AB, AC equal to the two sides DE, DF, each to each, namely, AB to DE, and AC to DF, and the angle BAC equal to the angle EDF: the base BC shall be equal to the base EF, and the triangle ABC to the triangle DEF, and the other angles shall be equal, each to each, to which the equal sides are opposite, namely, the angle ABC to the angle DEF, and the angle ACB to the angle DFE.



For if the triangle ABC be applied to the triangle DEF, so that the point A may be on the point D, and the straight line AB on the straight line DE, the point B will coincide with the point E, because AB is equal to DE. [*Hyp.*



And, AB coinciding with DE, AC will fall on DF, because the angle BAC is equal to the angle EDF. [*Hypothesis.*

Therefore also the point C will coincide with the point F, because AC is equal to DF. [*Hypothesis.*

But the point B was shewn to coincide with the point E, therefore the base BC will coincide with the base EF;

because, B coinciding with E and C with F, if the base BC does not coincide with the base EF, two straight lines will enclose a space; which is impossible. [*Axiom* 10.

Therefore the base BC coincides with the base EF, and is equal to it. [*Axiom* 8.

Therefore the whole triangle ABC coincides with the whole triangle DEF, and is equal to it. [*Axiom* 8.

And the other angles of the one coincide with the other angles of the other, and are equal to them, namely, the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

Wherefore, *if two triangles* &c. Q.E.D.

PROPOSITION 5. *THEOREM.*

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced the angles on the other side of the base shall be equal to one another.

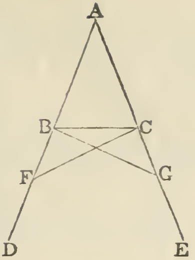
Let ABC be an isosceles triangle, having the side AB equal to the side AC, and let the straight lines AB, AC be produced to D and E: the angle ABC shall be equal to the angle ACB, and the angle CBD to the angle BCE.

In BD take any point F,

and from AE the greater cut off AG equal to AF the less, [I.3.

and join FC, GB.

Because AF is equal to AG, [*Constr.*
and AB to AC, [*Hypothesis.*
the two sides FA, AC are equal to the two sides GA, AB, each to each; and they contain the angle FAG common to the two triangles AFC, AGB;



therefore the base FC is equal to the base GB, and the triangle AFC to the triangle AGB, and the remaining angles of the one to the remaining angles of the other, each to each, to which the equal sides are opposite, namely the angle ACF to the angle ABG, and the angle AFC to the angle AGB. [I. 4.

And because the whole AF is equal to the whole AG, of which the parts AB, AC are equal, [*Hypothesis.*
the remainder BF is equal to the remainder CG. [*Axiom* 3.
And FC was shewn to be equal to GB;
therefore the two sides BF, FC are equal to the two sides CG, GB, each to each;
and the angle BFC was shewn to be equal to the angle CGB;
therefore the triangles BFC, CGB are equal, and their other angles are equal, each to each, to which the equal sides are opposite, namely the angle FBC to the angle GCB, and the angle BCF to the angle CBG. [I. 4.

And since it has been shewn that the whole angle ABG is equal to the whole angle ACF,
and that the parts of these, the angles CBG, BCF are also equal;
therefore the remaining angle ABC is equal to the remaining angle ACB, which are the angles at the base of the triangle ABC. [*Axiom* 3.

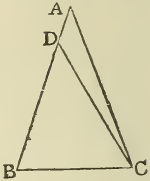
And it has also been shewn that the angle FBC is equal to the angle GCB, which are the angles on the other side of the base.

Wherefore, *the angles* &c. Q.E.D.

Corollary. Hence every equilateral triangle is also equiangular.

PROPOSITION 6. *THEOREM.*

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.



Let ABC be a triangle, having the angle ABC equal to the angle ACB: the side AC shall be equal to the side AB.

For if AC be not equal to AB, one of them must be greater than the other.

Let AB be the greater, and from it cut off DB equal to AC the less, [I. 3.
and join DC.

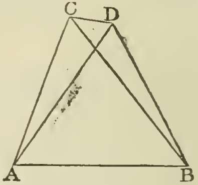
Then, because in the triangles DBC, ACB,
DB is equal to AC, [*Construction.*
and BC is common to both,
the two sides DB, BC are equal to the two sides AC, CB, each to each;
and the angle DBC is equal to the angle ACB; [*Hypothesis.*
therefore the base DC is equal to the base AB, and the triangle DBC is equal to the triangle ACB, [I. 4.
the less to the greater; which is absurd. [*Axiom* 9.
Therefore AB is not unequal to AC, that is, it is equal to it.

Wherefore, *if two angles* &c. Q.E.D.

Corollary. Hence every equiangular triangle is also equilateral.

PROPOSITION 7. *THEOREM.*

On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.



If it be possible, on the same base AB, and on the same side of it, let there be two triangles ACB, ADB, having their sides CA, DA, which are terminated at the extremity A of the base, equal

to one another, and likewise their sides CB, DB, which are terminated at B equal to one another.

Join CD. In the case in which the vertex of each triangle is without the other triangle;

because AC is equal to AD, [*Hypothesis.*

the angle ACD is equal to the angle ADC. [I. 5.

But the angle ACD is greater than the angle BCD, [*Ax.* 9.

therefore the angle ADC is also greater than the angle BCD;

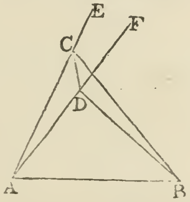
much more then is the angle BDC greater than the angle BCD.

Again, because BC is equal to BD, [*Hypothesis.*

the angle BDC is equal to the angle BCD. [I. 5.

But it has been shewn to be greater; which is impossible.

But if one of the vertices as D, be within the other triangle ACB, produce AC, AD to E, F.



Then because AC is equal to AD, in the triangle ACD, [*Hyp.*

the angles ECD, FDC, on the other side of the base CD, are equal to one another. [I. 5.

But the angle ECD is greater than the angle BCD, [*Axiom* 9.

therefore the angle FDC is also greater than the angle BCD;

much more then is the angle BDC greater than the angle BCD.

Again, because BC is equal to BD, [*Hypothesis.*

the angle BDC is equal to the angle BCD. [I. 5.

But it has been shewn to be greater; which is impossible.

The case in which the vertex of one triangle is on a side of the other needs no demonstration.

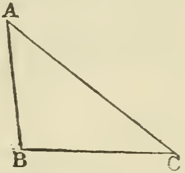
Wherefore, *on the same base* &c. Q.E.D.

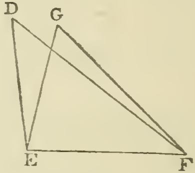
PROPOSITION 8. *THEOREM.*

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their

bases equal, the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides, equal to them, of the other.

Let ABC, DEF be two triangles, having the two sides AB, AC equal to the two sides DE, DF, each to each, namely AB to DE, and AC to DF, and also the base BC equal to the base EF: the angle BAC shall be equal to the angle EDF.





For if the triangle ABC be applied to the triangle DEF, so that the point B may be on the point E, and the straight line BC on the straight line EF, the point C will also coincide with the point F, because BC is equal to EF. [*Hyp.*
Therefore, BC coinciding with EF, BA and AC will coincide with ED and DF.

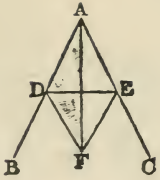
For if the base BC coincides with the base EF, but the sides BA, CA do not coincide with the sides ED, FD, but have a different situation as EG, FG; then on the same base and on the same side of it there will be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise their sides which are terminated at the other extremity.
But this is impossible. [I. 7.
Therefore since the base BC coincides with the base EF, the sides BA, AC must coincide with the sides ED, DF. Therefore also the angle BAC coincides with the angle EDF, and is equal to it. [*Axiom* 8.

Wherefore, *if two triangles* &c. Q.E.D.

PROPOSITION 9. *PROBLEM.*

To bisect a given rectilineal angle, that is to divide it into two equal angles.

Let BAC be the given rectilineal angle: it is required to bisect it.



Take any point D in AB, and from AC cut off AE equal to AD; [I. 3.

join DE, and on DE, on the side remote from A, describe the equilateral triangle DEF. [I. 1.

Join AF. The straight line AF shall bisect the angle BAC.

Because AD is equal to AE, [*Construction.*
and AF is common to the two triangles DAF, EAF,
the two sides DA, AF are equal to the two sides EA, AF, each to each;

and the base DF is equal to the base EF; [*Definition* 24.

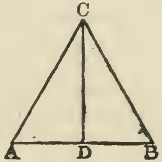
therefore the angle DAF is equal to the angle EAF. [I. 8.

Wherefore *the given rectilineal angle BAC is bisected by the straight line AF.* Q.E.F.

PROPOSITION 10. *PROBLEM.*

To bisect a given finite straight line, that is to divide it into two equal parts.

Let AB be the given straight line: it is required to divide it into two equal parts.



Describe on it an equilateral triangle ABC, [I. 1.

and bisect the angle ACB by the straight line CD, meeting AB at D. [I. 9.

AB shall be cut into two equal parts at the point D.

Because AC is equal to CB, [*Definition* 24.

and CD is common to the two triangles ACD, BCD,

the two sides AC, CD are equal to the two sides BC, CD, each to each;

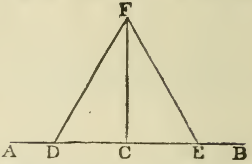
and the angle ACD is equal to the angle BCD; [*Constr.*

therefore the base AD is equal to the base DB. [I. 4.

Wherefore *the given straight line AB is divided into two equal parts at the point D.* Q.E.F.

PROPOSITION 11. *PROBLEM.*

To draw a straight line at right angles to a given straight line, from a given point in the same.



Let AB be the given straight line, and C the given point in it: it is required to draw from the point C a straight line at right angles to AB.

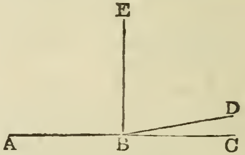
Take any point D in AC, and make CE equal to CD. [I. 3.
On DE describe the equilateral triangle DFE, [I. 1.
and join CF.

The straight line CF drawn from the given point C shall be at right angles to the given straight line AB.

Because DC is equal to CE, [*Construction.*
and CF is common to the two triangles DCF, ECF;
the two sides DC, CF are equal to the two sides EC, CF, each to each;
and the base DF is equal to the base EF; [*Definition* 24.
therefore the angle DCF is equal to the angle ECF; [I. 8.
and they are adjacent angles.
But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle; [*Definition* 10.
therefore each of the angles DCF, ECF is a right angle.

Wherefore *from the given point C in the given straight line AB, CF has been drawn at right angles to AB.* Q.E.F.

Corollary. By the help of this problem it may be shewn that two straight lines cannot have a common segment.



If it be possible, let the two straight lines ABC, ABD have the segment AB common to both of them.

From the point B draw BE at right angles to AB.

Then, because ABC is a straight line, [*Hypothesis.*
the angle CBE is equal to the angle EBA. [*Definition* 10.

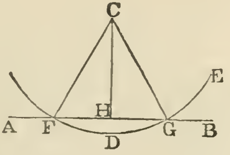
Also, because ABD is a straight line, [*Hypothesis.*
the angle DBE is equal to the angle EBA.
Therefore the angle DBE is equal to the angle CBE, [*Ax.* 1.
the less to the greater; which is impossible. [*Axiom* 9.

Wherefore *two straight lines cannot have a common segment.*

PROPOSITION 12. *PROBLEM.*

To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced to any length both ways, and let C be the given point without it: it is required to draw from the point C a straight line perpendicular to AB.



Take any point D on the other side of AB, and from the centre C, at the distance CD, describe the circle EGF, meeting AB at F and G. [*Postulate* 3.
Bisect FG at H, [I. 10.
and join CH.

The straight line CH drawn from the given point C shall be perpendicular to the given straight line AB.

Join CF, CG.

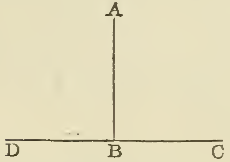
Because FH is equal to HG, [*Construction.*
and HC is common to the two triangles FHC, GHC;
the two sides FH, HC are equal to the two sides GH, HC, each to each;
and the base CF is equal to the base CG; [*Definition* 15.
therefore the angle CHF is equal to the angle CHG; [I. 8.
and they are adjacent angles.
But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it. [*Def.* 10.

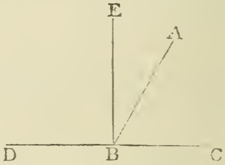
Wherefore *a perpendicular CH has been drawn to the given straight line AB from the given point C without it.* Q.E.F.

PROPOSITION 13. *THEOREM.*

The angles which one straight line makes with another straight line on one side of it, either are two right angles, or are together equal to two right angles.

Let the straight line *AB* make with the straight line *CD*, on one side of it, the angles *CBA*, *ABD*: these either are two right angles, or are together equal to two right angles.





For if the angle *CBA* is equal to the angle *ABD*, each of them is a right angle. [*Definition* 10.

But if not, from the point *B* draw *BE* at right angles to *CD*; [I. 11.

therefore the angles *CBE, EBD* are two right angles.[*Def.*10.

Now the angle *CBE* is equal to the two angles *CBA, ABE*; to each of these equals add the angle *EBD*;

therefore the angles *CBE, EBD* are equal to the three angles *CBA, ABE, EBD.* [*Axiom* 2.

Again, the angle *DBA* is equal to the two angles *DBE, EBA*;

to each of these equals add the angle *ABC*;

therefore the angles *DBA, ABC* are equal to the three angles *DBE, EBA, ABC.* [*Axiom* 2.

But the angles *CBE, EBD* have been shewn to be equal to the same three angles.

Therefore the angles *CBE, EBD* are equal to the angles *DBA, ABC.* [*Axiom* 1.

But *CBE, EBD* are two right angles;

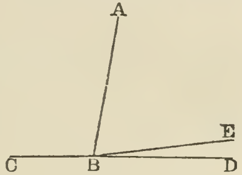
therefore *DBA, ABC* are together equal to two right angles.

Wherefore, *the angles* &c. Q.E.D.

PROPOSITION 14. *THEOREM.*

If, at a point in a straight line, two other straight lines, on the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

At the point *B* in the straight line *AB*, let the two straight lines *BC, BD*, on the opposite sides of *AB*, make the adjacent angles *ABC, ABD* together equal to two right angles: *BD* shall be in the same straight line with *CB*.



For if *BD* be not in the same straight line with *CB*, let *BE* be in the same straight line with it.

Then because the straight line *AB* makes with the straight line *CBE*, on one side of it, the angles *ABC, ABE*, these angles are together equal to two right angles. [I. 13.

But the angles *ABC, ABD* are also together equal to two right angles. [*Hypothesis.*

Therefore the angles *ABC, ABE* are equal to the angles *ABC, ABD*.

From each of these equals take away the common angle *ABC*, and the remaining angle *ABE* is equal to the remaining angle *ABD*, [*Axiom* 3.

the less to the greater; which is impossible.

Therefore *BE* is not in the same straight line with *CB*.

And in the same manner it may be shewn that no other can be in the same straight line with it but *BD*;

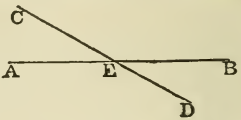
therefore *BD* is in the same straight line with *CB*.

Wherefore, *if at a point* &c. Q.E.D.

PROPOSITION 15. *THEOREM.*

If two straight lines cut one another, the vertical, or opposite, angles shall be equal.

Let the two straight lines *AB*, *CD* cut one another at the point *E*; the angle *AEC* shall be equal to the angle *DEB*, and the angle *CEB* to the angle *AED*.



Because the straight line *AE* makes with the straight line *CD* the angles *CEA*, *AED*, these angles are together equal to two right angles. [I. 13.

Again, because the straight line *DE* makes with the straight line *AB* the angles *AED*, *DEB*, these also are together equal to two right angles. [I. 13.

But the angles *CEA*, *AED* have been shewn to be together equal to two right angles.

Therefore the angles *CEA*, *AED* are equal to the angles *AED*, *DEB*.

From each of these equals take away the common angle *AED*, and the remaining angle *CEA* is equal to the remaining angle *DEB*. [*Axiom* 3.

In the same manner it may be shewn that the angle *CEB* is equal to the angle *AED*.

Wherefore, *if two straight lines* &c. Q.E.D.

Corollary 1. From this it is manifest that, if two straight lines cut one another, the angles which they make at the point where they cut, are together equal to four right angles.

Corollary 2. And consequently, that all the angles made by any number of straight lines meeting at one point, are together equal to four right angles.

PROPOSITION 16. *THEOREM.*

If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

Let *ABC* be a triangle, and let one side *BC* be produced to *D*: the exterior angle *ACD* shall be greater than either of the interior opposite angles *CBA*, *BAC*.

Bisect *AC* at *E*, [I. 10.

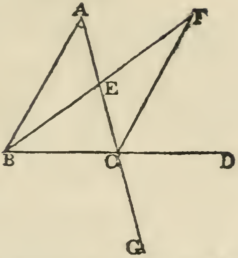
join *BE* and produce it to *F*, making *EF* equal to *EB*, [I. 3.

and join *FC*.

Because *AE* is equal to *EC*, and *BE* to *EF*; [*Constr.*

the two sides *AE*, *EB* are equal to the two sides *CE*, *EF* each to each;

and the angle *AEB* is equal to the angle *CEF*, because they are opposite vertical angles; [I. 15.



therefore the triangle *AEB* is equal to the triangle *CEF*, and the remaining angles to the remaining angles, each to each, to which the equal sides are opposite; [I. 4.

therefore the angle *BAE* is equal to the angle *ECF*.

But the angle *ECD* is greater than the angle *ECF*. [*Axiom* 9.

Therefore the angle *ACD* is greater than the angle *BAE*.

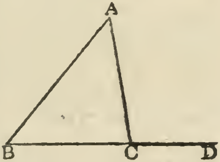
In the same manner if *BC* be bisected, and the side *AC* be produced to *G*, it may be shewn that the angle *BCG*, that is the angle *ACD*, is greater than the angle *ABC*. [I. 15.

Wherefore, *if one side* &c. Q.E.D.

PROPOSITION 17. *THEOREM.*

Any two angles of a triangle are together less than two right angles.

Let *ABC* be a triangle: any two of its angles are together less than two right angles.



Produce *BC* to *D*.

Then because *ACD* is the exterior angle of the triangle *ABC*, it is greater than the interior opposite angle *ABC*. [I. 16.

To each of these add the angle *ACB*.

Therefore the angles *ACD*, *ACB* are greater than the angles *ABC*, *ACB*.

But the angles *ACD*, *ACB* are together equal to two right angles. [I. 13.

Therefore the angles *ABC*, *ACB* are together less than two right angles.

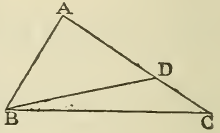
In the same manner it may be shewn that the angles *BAC*, *ACB*, as also the angles *CAB*, *ABC*, are together less than two right angles.

Wherefore, *any two angles* &c. Q.E.D.

PROPOSITION 18. *THEOREM.*

The greater side of every triangle has the greater angle opposite to it.

Let ABC be a triangle, of which the side AC is greater than the side AB: the angle ABC is also greater than the angle ACB.



Because AC is greater than AB, make AD equal to AB, [I. 3. and join BD.

Then, because ADB is the exterior angle of the triangle BDC, it is greater than the interior opposite angle DCB. [I. 16.

But the angle ADB is equal to the angle ABD, [I. 5.
because the side AD is equal to the side AB. [*Constr.*

Therefore the angle ABD is also greater than the angle ACB.

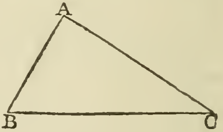
Much more then is the angle ABC greater than the angle ACB. [*Axiom* 9.

Wherefore, *the greater side* &c. Q.E.D.

PROPOSITION 19. *THEOREM.*

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Let ABC be a triangle, of which the angle ABC is greater than the angle ACB: the side AC is also greater than the side AB.



For if not, AC must be either equal to AB or less than AB.

But AC is not equal to AB,
for then the angle ABC would be equal to the angle ACB; [I. 5.
but it is not; [*Hypothesis.*
therefore AC is not equal to AB.

Neither is AC less than AB,
for then the angle ABC would be less than the angle ACB; [I. 18.
but it is not; [*Hypothesis.*

therefore AC is not less than AB.

And it has been shewn that AC is not equal to AB.

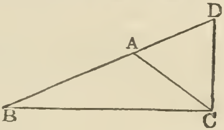
Therefore AC is greater than AB.

Wherefore, *the greater angle* &c. Q.E.D.

PROPOSITION 20. *THEOREM.*

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle: any two sides of it are together greater than the third side; namely, BA, AC greater than BC; and AB, BC greater than AC; and BC, CA greater than AB.



Produce BA to D, making AD equal to AC, [I. 3.
and join DC.

Then, because AD is equal to AC, [*Construction.*
the angle ADC is equal to the angle ACD. [I. 5.
But the angle BCD is greater than the angle ACD. [*Ax.* 9.
Therefore the angle BCD is greater than the angle BDC.
And because the angle BCD of the triangle BCD is greater than its angle BDC, and that the greater angle is subtended by the greater side; [I. 19.
therefore the side BD is greater than the side BC.
But BD is equal to BA and AC.
Therefore BA, AC are greater than BC.

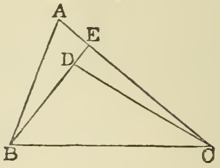
In the same manner it may be shewn that AB, BC are greater than AC, and BC, CA greater than AB.

Wherefore, *any two sides* &c. Q.E.D.

PROPOSITION 21. *THEOREM.*

If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let ABC be a triangle, and from the points B, C, the ends of the side BC, let the two straight lines BD, CD be drawn to the point D within the triangle: BD, DC shall be less than the other two sides BA, AC of the triangle, but shall contain an angle BDC greater than the angle BAC.



Produce BD to meet AC at E.

Because two sides of a triangle are greater than the third side, the two sides BA, AE of the triangle ABE are greater than the side BE. [I. 20.

To each of these add EC.

Therefore BA, AC are greater than BE, EC.

Again; the two sides CE, ED of the triangle CED are greater than the third side CD. [I. 20.

To each of these add DB.

Therefore CE, EB are greater than CD, DB.

But it has been shewn that BA, AC are greater than BE, EC;

much more then are BA, AC greater than BD, DC.

Again, because the exterior angle of any triangle is greater than the interior opposite angle, the exterior angle BDC of the triangle CDE is greater than the angle CED. [I. 16.

For the same reason, the exterior angle CEB of the triangle ABE is greater than the angle BAE.

But it has been shewn that the angle BDC is greater than the angle CEB;

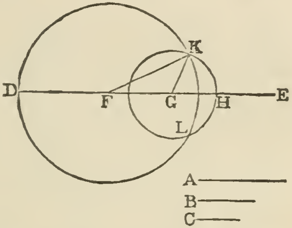
much more then is the angle BDC greater than the angle BAC.

Wherefore, *if from the ends* &c. Q.E.D.

PROPOSITION 22. *PROBLEM.*

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.

Let A, B, C be the three given straight lines, of which any two whatever are greater than the third; namely, A and B greater than C; A and C greater than B; and B and C greater than A: it is required to make a triangle of which the sides shall be equal to A, B, C, each to each.



Take a straight line DE terminated at the point D, but unlimited towards E, and make DF equal to A, FG equal to B, and GH equal to C. [I. 3.

From the centre F, at the distance FD, describe the circle DKL. [*Post.* 3.

From the centre G, at the distance GH, describe the circle HLK, cutting the former circle at K.

Join KF, KG. The triangle KFG shall have its sides equal to the three straight lines A, B, C.

Because the point F is the centre of the circle DKL, FD is equal to FK. [*Definition* 15.

But FD is equal to A. [*Construction.*

Therefore FK is equal to A. [*Axiom* 1.

Again, because the point G is the centre of the circle HLK, GH is equal to GK. [*Definition* 15.

But GH is equal to C. [*Construction.*

Therefore GK is equal to C. [*Axiom* 1.

And FG is equal to B. [*Construction.*

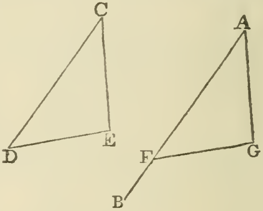
Therefore the three straight lines KF, FG, GK are equal to the three A, B, C.

Wherefore *the triangle KFG has its three sides KF, FG, GK equal to the three given straight lines A, B, C.* Q.E.F.

PROPOSITION 23. *PROBLEM.*

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle: it is required to make at the given point A, in the given straight line AB, an angle equal to the given rectilineal angle DCE.



In CD, CE take any points D, E, and join DE. Make the triangle AFG the sides of which shall be equal to the three straight lines CD, DE, EC; so that AF shall be equal to CD, AG to CE, and FG to DE. [I. 22.

The angle FAG shall be equal to the angle DCE.

Because FA, AG are equal to DC, CE, each to each, and the base FG equal to the base DE; [*Construction.*

therefore the angle FAG is equal to the angle DCE. [I. 8.

Wherefore *at the given point A in the given straight line AB, the angle FAG has been made equal to the given rectilineal angle DCE.* Q.E.F.

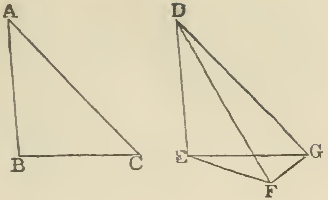
PROPOSITION 24. *THEOREM.*

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them, of the other, the base of that which has the greater angle shall be greater than the base of the other.

Let ABC, DEF be two triangles, which have the two sides AB, AC, equal to the two sides DE, DF, each to each, namely, AB to DE, and AC to DF, but the angle BAC greater than the angle EDF: the base BC shall be

greater than the base EF.

Of the two sides DE, DF, let DE be the side which is not greater than the other. At the point D in the straight line DE, make the angle EDG equal to the angle BAC, [I. 23.



and make DG equal to AC or DF, [I. 3.
and join EG, GF.

Because AB is equal to DE, [*Hypothesis.*
and AC to DG; [*Construction.*
the two sides BA, AC are equal to the two sides ED, DG, each to each;
and the angle BAC is equal to the angle EDG; [*Constr.*
therefore the base BC is equal to the base EG. [I. 4.

And because DG is equal to DF, [*Construction.*
the angle DGF is equal to the angle DFG. [I. 5.
But the angle DGF is greater than the angle EGF. [*Ax.* 9.
Therefore the angle DFG is greater than the angle EGF.
Much more then is the angle EFG greater than the angle EGF. [*Axiom* 9.

And because the angle EFG of the triangle EFG is greater than its angle EGF, and that the greater angle is subtended by the greater side, [I. 19.
therefore the side EG is greater than the side EF.
But EG was shewn to be equal to BC;
therefore BC is greater than EF.

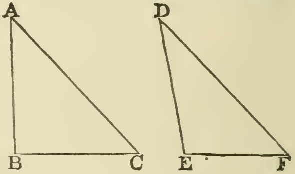
Wherefore, *if two triangles* &c. Q.E.D.

PROPOSITION 25. *THEOREM.*

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one

greater than the base of the other, the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the sides equal to them, of the other.

Let ABC, DEF be two triangles, which have the two sides AB, AC equal to the two sides DE, DF, each to each, namely, AB to DE, and AC to DF, but the base BC greater than the base EF: the angle BAC shall be greater than the angle EDF.



For if not, the angle BAC must be either equal to the angle EDF or less than the angle EDF.

But the angle BAC is not equal to the angle EDF, for then the base BC would be equal to the base EF; [I. 4.
but it is not; [*Hypothesis.*
therefore the angle BAC is not equal to the angle EDF.
Neither is the angle BAC less than the angle EDF,
for then the base BC would be less than the base EF; [I. 24.
but it is not; [*Hypothesis.*
therefore the angle BAC is not less than the angle EDF.
And it has been shewn that the angle BAC is not equal to the angle EDF.
Therefore the angle BAC is greater than the angle EDF.

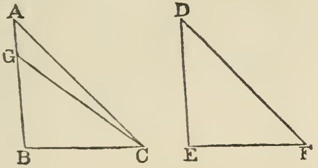
Wherefore, *if two triangles* &c. Q.E.D.

PROPOSITION 26. *THEOREM.*

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, either the sides adjacent to the equal angles, or sides which are opposite to equal angles in each, then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.

Let ABC, DEF be two triangles, which have the angles ABC, BCA equal to the angles DEF, EFD, each

to each, namely, ABC to DEF, and BCA to EFD; and let them have also one side equal to one side; and first let those sides be equal which are adjacent to the equal angles in the two triangles, namely, BC to EF: the other sides shall be equal, each to each, namely, AB to DE, and AC to DF, and the third angle BAC equal to the third angle EDF.



For if AB be not equal to DE, one of them must be greater than the other. Let AB be the greater, and make BG equal to DE, [I. 3.

and join GC.

Then because GB is equal to DE, [*Construction.*

and BC to EF; [*Hypothesis.*

the two sides GB, BC are equal to the two sides DE, EF, each to each;

and the angle GBC is equal to the angle DEF; [*Hypothesis.*

therefore the triangle GBC is equal to the triangle DEF, and the other angles to the other angles, each to each, to which the equal sides are opposite; [I. 4.

therefore the angle GCB is equal to the angle DFE.

But the angle DFE is equal to the angle ACB. [*Hypothesis.*

Therefore the angle GCB is equal to the angle ACB, [*Ax.* 1.

the less to the greater; which is impossible.

Therefore AB is not unequal to DE,

that is, it is equal to it;

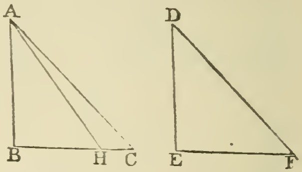
and BC is equal to EF; [*Hypothesis.*

therefore the two sides AB, BC are equal to the two sides DE, EF, each to each;

and the angle ABC is equal to the angle DEF; [*Hypothesis.*

therefore the base AC is equal to the base DF, and the third angle BAC to the third angle EDF. [I. 4.

Next, let sides which are opposite to equal angles in each triangle be equal to one another, namely, AB to DE: likewise in this case the other sides shall be equal, each to each, namely, BC to EF, and AC to DF, and also the third angle BAC equal to the third angle EDF.



For if BC be not equal to EF, one of them must be greater than the other. Let BC be the greater, and make BH equal to EF, [I. 3.
and join AH.

Then because BH is equal to EF, [*Construction.*
and AB to DE; [*Hypothesis.*
the two sides AB, BH are equal to the two sides DE, EF, each to each;
and the angle ABH is equal to the angle DEF; [*Hypothesis.*
therefore the triangle ABH is equal to the triangle DEF, and the other angles to the other angles, each to each, to which the equal sides are opposite; [I. 4.
therefore the angle BHA is equal to the angle EFD.
But the angle EFD is equal to the angle BCA. [*Hypothesis.*
Therefore the angle BHA is equal to the angle BCA; [*Ax.* 1.
that is, the exterior angle BHA of the triangle AHC is equal to its interior opposite angle BCA;
which is impossible. [I. 16.
Therefore BC is not unequal to EF,
that is, it is equal to it;
and AB is equal to DE; [*Hypothesis.*
therefore the two sides AB, BC are equal to the two sides DE, EF, each to each;
and the angle ABC is equal to the angle DEF; [*Hypothesis.*
therefore the base AC is equal to the base DF, and the third angle BAC to the third angle EDF. [I. 4.

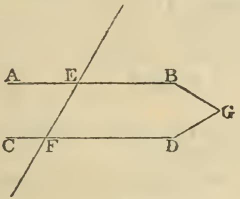
Wherefore, *if two triangles* &c. Q.E.D.

PROPOSITION 27. *THEOREM.*

If a straight line falling on two other straight lines, make the alternate angles equal to one another, the two straight lines shall be parallel to one another.

Let the straight line EF, which falls on the two straight lines AB, CD, make the alternate angles AEF, EFD equal to one another: AB shall be parallel to CD.

For if not, AB and CD, being produced, will meet either towards B, D or towards A, C. Let them be produced and meet towards B, D at the point G.



Therefore GEF is a triangle, and its exterior angle AEF is greater than the interior opposite angle EFG; [I. 16.

But the angle AEF is also equal to the angle EFG; [*Hyp.* which is impossible.

Therefore AB and CD being produced, do not meet towards B, D.

In the same manner, it may be shewn that they do not meet towards A, C.

But those straight lines which being produced ever so far both ways do not meet, are parallel. [*Definition* 35.

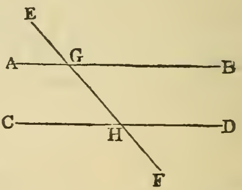
Therefore AB is parallel to CD.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 28. *THEOREM.*

If a straight line falling on two other straight lines, make the exterior angle equal to the interior and opposite angle on the same side of the line, or make the interior angles on the same side together equal to two right angles, the two straight lines shall be parallel to one another.

Let the straight line *EF*, which falls on the two straight lines *AB*, *CD*, make the exterior angle *EGB* equal to the interior and opposite angle *GHD* on the same side, or make the interior angles on the same side *BGH*, *GHD* together equal to two right angles: *AB* shall be parallel to *CD*.



Because the angle *EGB* is equal to the angle *GHD*, [*Hyp.*
and the angle *EGB* is also equal to the angle *AGH*, [I. 15.
therefore the angle *AGH* is equal to the angle *GHD*; [*Ax.* 1.
and they are alternate angles;
therefore *AB* is parallel to *CD*. [I. 27.

Again; because the angles *BGH*, *GHD* are together equal to two right angles, [*Hypothesis.*
and the angles *AGH*, *BGH* are also together equal to two right angles, [I. 13.
therefore the angles *AGH*, *BGH* are equal to the angles *BGH*, *GHD*.
Take away the common angle *BGH*; therefore the remaining angle *AGH* is equal to the remaining angle *GHD*; [*Axiom* 3.
and they are alternate angles;
therefore *AB* is parallel to *CD*. [I. 27.

Wherefore, *if a straight line* &c. Q.E.D.

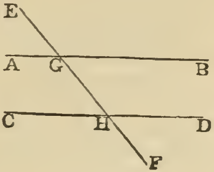
PROPOSITION 29. *THEOREM.*

If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

Let the straight line *EF* fall on the two parallel straight lines *AB*, *CD*: the alternate angles *AGH*, *GHD* shall be equal to one another, and the exterior angle *EGB* shall be equal to the interior and opposite angle

on the same side, *GHD*, and the two interior angles on the same side, *BGH*, *GHD*, shall be together equal to two right angles.

For if the angle *AGH* be not equal to the angle *GHD*, one of them must be greater than the other; let the angle *AGH* be the greater.



Then the angle *AGH* is greater than the angle *GHD*;

to each of them add the angle *BGH*;

therefore the angles *AGH*, *BGH* are greater than the angles *BGH*, *GHD*.

But the angles *AGH*, *BGH* are together equal to two right angles; [I. 13.

therefore the angles *BGH*, *GHD* are together less than two right angles.

But if a straight line meet two straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines being continually produced, shall at length meet on that side on which are the angles which are less than two right angles. [*Axiom* 12.

Therefore the straight lines *AB*, *CD*, if continually produced, will meet.

But they never meet, since they are parallel by hypothesis.

Therefore the angle *AGH* is not unequal to the angle *GHD*; that is, it is equal to it.

But the angle *AGH* is equal to the angle *EGB*. [I. 15.

Therefore the angle *EGB* is equal to the angle *GHD*. [*Ax.* 1.

Add to each of these the angle *BGH*.

Therefore the angles *EGB*, *BGH* are equal to the angles *BGH*, *GHD*. [*Axiom* 2.

But the angles *EGB*, *BGH* are together equal to two right angles. [I. 13.

Therefore the angles *BGH*, *GHD* are together equal to two right angles. [*Axiom* 1.

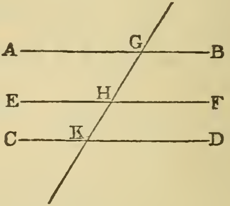
Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 30. *THEOREM.*

Straight lines which are parallel to the same straight line are parallel to each other.

Let AB, CD be each of them parallel to EF: AB shall be parallel to CD.

Let the straight line GHK cut AB, EF, CD.



Then, because GHK cuts the parallel straight lines AB, EF, the angle AGH is equal to the angle GHF. [I. 29.

Again, because GK cuts the parallel straight lines EF, CD, the angle GHF is equal to the angle GKD. [I. 29.

And it was shewn that the angle AGK is equal to the angle GHF.

Therefore the angle AGK is equal to the angle GKD; [*Ax.* 1.

and they are alternate angles;

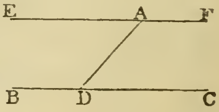
therefore AB is parallel to CD [I. 27.

Wherefore, *straight lines* &c. Q.E.D.

PROPOSITION 31. *PROBLEM.*

To draw a straight line through a given point parallel to a given straight line.

Let A be the given point, and BC the given straight line: it is required to draw a straight line through the point A parallel to the straight line BC.



In BC take any point D, and join AD; at the point A in the straight line AD, make the angle DAE equal to the angle ADC; [I. 23.

and produce the straight line EA to F.

EF shall be parallel to BC.

Because the straight line AD, which meets the two straight lines BC, EF, makes the alternate angles EAD, ADC equal to one another, [*Construction.*

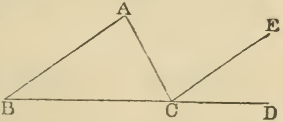
EF is parallel to BC. [I. 27.

Wherefore *the straight line* EAF *is drawn through the given point* A, *parallel to the given straight line* BC. Q.E.F.

PROPOSITION 32. *THEOREM.*

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

Let ABC be a triangle, and let one of its sides BC be produced to D: the exterior angle ACD shall be equal to the two interior and opposite angles CAB, ABC; and the three interior angles of the triangle, namely, ABC, BCA, CAB shall be equal to two right angles.



Through the point C draw CE parallel to AB. [I. 31.

Then, because AB is parallel to CE, and AC falls on them, the alternate angles BAC, ACE are equal. [I. 29.

Again, because AB is parallel to CE, and BD falls on them, the exterior angle ECD is equal to the interior and opposite angle ABC. [I. 29.

But the angle ACE was shewn to be equal to the angle BAC;

therefore the whole exterior angle ACD is equal to the two interior and opposite angles CAB, ABC. [*Axiom* 2.

To each of these equals add the angle ACB;

therefore the angles ACD, ACB are equal to the three angles CBA, BAC, ACB. [*Axiom* 2.

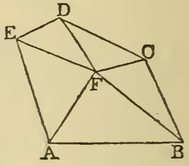
But the angles ACD, ACB are together equal to two right angles; [I. 13.

therefore also the angles CBA, BAC, ACB are together equal to two right angles. [*Axiom* 1.

Wherefore, *if a side of any triangle* &c. Q.E.D.

COROLLARY 1. *All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*

For any rectilineal figure $ABCDE$ can be divided into as many triangles as the figure has sides, by drawing straight lines from a point F within the figure to each of its angles.

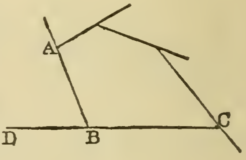


And by the preceding proposition, all the angles of these triangles are equal to twice as many right angles as there are triangles, that is, as the figure has sides.

And the same angles are equal to the interior angles of the figure, together with the angles at the point F, which is the common vertex of the triangles, that is, together with four right angles. [I. 15. *Corollary* 2.

Therefore all the interior angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

COROLLARY 2. *All the exterior angles of any rectilineal figure are together equal to four right angles.*



Because every interior angle ABC, with its adjacent exterior angle ABD, is equal to two right angles; [I. 13.

therefore all the interior angles of the figure, together with all its exterior angles, are equal to twice as many right angles as the figure has sides.

But, by the foregoing Corollary all the interior angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

Therefore all the interior angles of the figure, together with all its exterior angles, are equal to all the interior angles of the figure, together with four right angles.

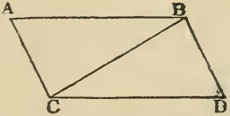
Therefore all the exterior angles are equal to four right angles.

PROPOSITION 33. *THEOREM.*

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.

Let AB and CD be equal and parallel straight lines, and let them be joined towards the same parts by the straight lines AC and BD: AC and BD shall be equal and parallel.

Join BC.



Then because AB is parallel to CD, [*Hypothesis.*
and BC meets them,
the alternate angles ABC, BCD are equal. [I. 29.

And because AB is equal to CD, [*Hypothesis.*
and BC is common to the two triangles ABC, DCB; the two sides AB, BC are equal to the two sides DC, CB, each to each;
and the angle ABC was shewn to be equal to the angle BCD;
therefore the base AC is equal to the base BD, and the triangle ABC to the triangle BCD, and the other angles to the other angles, each to each, to which the equal sides are opposite; [I. 4.
therefore the angle ACB is equal to the angle CBD.

And because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD equal to one another, AC is parallel to BD. [I. 27.
And it was shewn to be equal to it.

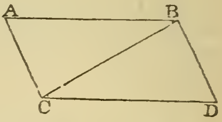
Wherefore, *the straight lines* &c. Q.E.D.

PROPOSITION 34. *THEOREM.*

The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects the parallelogram, that is, divides it into two equal parts.

Note. A parallelogram is a four-sided figure of which the opposite sides are parallel; and a diameter is the straight line joining two of its opposite angles.

Let $ACDB$ be a parallelogram, of which BC is a diameter; the opposite sides and angles of the figure shall be equal to one another, and the diameter BC shall bisect it.



Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal to one another. [I. 29.

And because AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are equal to one another. [I. 29.

Therefore the two triangles ABC, BCD have two angles ABC, BCA in the one, equal to two angles DCB, CBD in the other, each to each, and one side BC is common to the two triangles, which is adjacent to their equal angles;

therefore their other sides are equal, each to each, and the third angle of the one to the third angle of the other, namely, the side AB equal to the side CD, and the side AC equal to the side BD, and the angle BAC equal to the angle CDB. [I. 26.

And because the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB,

the whole angle ABD is equal to the whole angle ACD. [*Ax.* 2.

And the angle BAC has been shewn to be equal to the angle CDB.

Therefore the opposite sides and angles of a parallelogram are equal to one another.

Also the diameter bisects the parallelogram.

For AB being equal to CD, and BC common,

the two sides AB, BC are equal to the two sides DC, CB each to each;

and the angle ABC has been shewn to be equal to the angle BCD;

therefore the triangle ABC is equal to the triangle BCD, [I. 4.

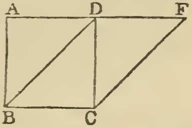
and the diameter BC divides the parallelogram $ACDB$ into two equal parts.

Wherefore, *the opposite sides* &c. Q.E.D.

PROPOSITION 35. *THEOREM.*

Parallelograms on the same base, and between the same parallels, are equal to one another.

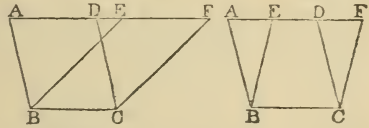
Let the parallelograms $ABCD$, $EBCF$ be on the same base BC, and between the same parallels AF, BC: the parallelogram $ABCD$ shall be equal to the parallelogram $EBCF$.



If the sides AD, DF of the parallelograms $ABCD$, $DBCF$, opposite to the base BC, be terminated at the same point D, it is plain that each of the parallelograms is double of the triangle BDC: [I. 34.

and they are therefore equal to one another. [*Axiom* 6.

But if the sides AD, EF, opposite to the base BC of the parallelograms $ABCD$, $EBCF$ be not terminated at the same point, then, because $ABCD$ is a parallelogram AD is equal to BC; [I. 34.



for the same reason EF is equal to BC;

therefore AD is equal to EF; [*Axiom* 1.

therefore the whole, or the remainder, AE is equal to the whole, or the remainder, DF. [*Axioms* 2, 3.

And AB is equal to DC; [I. 34.

therefore the two sides EA, AB are equal to the two sides FD, DC each to each;

and the exterior angle FDC is equal to the interior and opposite angle EAB; [I. 29.

therefore the triangle EAB is equal to the triangle FDC. [I. 4.

Take the triangle FDC from the trapezium $ABCF$, and from the same trapezium take the triangle EAB, and the remainders are equal; [*Axiom* 3.

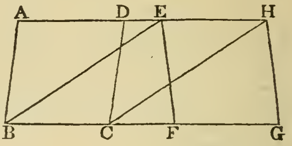
that is, the parallelogram $ABCD$ is equal to the parallelogram $EBCF$.

Wherefore, *parallelograms on the same base* &c. Q.E.D.

PROPOSITION 36. *THEOREM.*

Parallelograms on equal bases, and between the same parallels, are equal to one another.

Let *ABCD, EFGH* be parallelograms on equal bases *BC, FG*, and between the same parallels *AH, BG*: the parallelogram *ABCD* shall be equal to the parallelogram *EFGH*.



Join *BE, CH.*

Then, because *BC* is equal to *FG*, [*Hyp.*
and *FG* to *EH*, [I. 34.
BC is equal to *EH*; [*Axiom* 1.
and they are parallels, [*Hypothesis.*
and joined towards the same parts by the straight lines *BE, CH.*

But straight lines which join the extremities of equal and parallel straight lines towards the same parts are themselves equal and parallel. [I. 33.

Therefore *BE, CH* are both equal and parallel.

Therefore *EBCH* is a parallelogram. [*Definition.*

And it is equal to *ABCD*, because they are on the same base *BC*, and between the same parallels *BC, AH*. [I. 35.

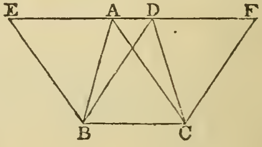
For the same reason the parallelogram *EFGH* is equal to the same *EBCH*.

Therefore the parallelogram *ABCD* is equal to the parallelogram *EFGH*. [*Axiom* 1.

Wherefore, *parallelograms* &c. Q.E.D.

PROPOSITION 37. *THEOREM.*

Triangles on the same base, and between the same parallels, are equal.



Let the triangles *ABC, DBC* be on the same base *BC*, and between the same parallels *AD, BC*: the triangle *ABC* shall be equal to the triangle *DBC*.

Produce *AD* both ways to the points *E, F*; [*Post.* 2.

through B draw BE parallel to CA, and through C draw CF parallel to BD. [I. 31.

Then each of the figures $EBCA$, $DBCF$ is a parallelogram; [*Definition.*

and $EBCA$ is equal to $DBCF$, because they are on the same base BC, and between the same parallels BC, EF. [I. 35.

And the triangle ABC is half of the parallelogram $EBCA$, because the diameter AB bisects the parallelogram; [I. 34.

and the triangle DBC is half of the parallelogram $DBCF$, because the diameter DC bisects the parallelogram. [I. 34.

But the halves of equal things are equal. [*Axiom* 7.

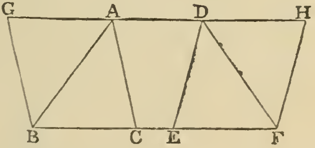
Therefore the triangle ABC is equal to the triangle DBC.

Wherefore, *triangles* &c. Q.E.D.

PROPOSITION 38. *THEOREM.*

Triangles on equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC, DEF be on equal bases BC, EF, and between the same parallels BF, AD: the triangle ABC shall be equal to the triangle DEF.



Produce AD both ways to the points G, H; through B draw BG parallel to CA, and through F draw FH parallel to ED. [I. 31.

Then each of the figures $GBCA$, $DEFH$ is a parallelogram. [*Definition.*

And they are equal to one another because they are on equal bases BC, EF, and between the same parallels BF, GH. [I. 36.

And the triangle ABC is half of the parallelogram $GBCA$, because the diameter AB bisects the parallelogram; [I. 34.

and the triangle DEF is half of the parallelogram $DEFH$, because the diameter DF bisects the parallelogram.

But the halves of equal things are equal. [*Axiom* 7.

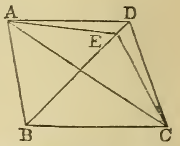
Therefore the triangle ABC is equal to the triangle DEF.

Wherefore, *triangles* &c. Q.E.D.

PROPOSITION 39. *THEOREM.*

Equal triangles on the same base, and on the same side of it, are between the same parallels.

Let the equal triangles ABC, DBC be on the same base BC, and on the same side of it: they shall be between the same parallels.



Join AD.

AD shall be parallel to BC.

For if it is not, through A draw AE parallel to BC, meeting BD at E. [I. 31.

and join EC.

Then the triangle ABC is equal to the triangle EBC, because they are on the same base BC, and between the same parallels BC, AE. [I. 37.

But the triangle ABC is equal to the triangle DBC. [*Hyp.*

Therefore also the triangle DBC is equal to the triangle EBC, [*Axiom* 1.

the greater to the less; which is impossible.

Therefore AE is not parallel to BC.

In the same manner it can be shewn, that no other straight line through A but AD is parallel to BC;

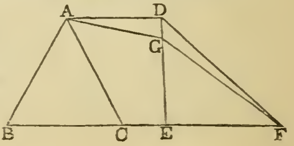
therefore AD is parallel to BC.

Wherefore, *equal triangles* &c. Q.E.D.

PROPOSITION 40. *THEOREM.*

Equal triangles, on equal bases, in the same straight line, and on the same side of it, are between the same parallels.

Let the equal triangles ABC, DEF be on equal bases BC, EF, in the same straight line BF, and on the same side of it: they shall be between the same parallels.



Join AD.

AD shall be parallel to BF.

For if it is not, through A draw AG parallel to BF, meeting ED at G [I. 31.

and join GF.

Then the triangle ABC is equal to the triangle GEF, because they are on equal bases BC, EF, and between the same parallels. [I. 38.

But the triangle ABC is equal to the triangle DEF. [*Hyp.*

Therefore also the triangle DEF is equal to the triangle GEF, [*Axiom* 1.

the greater to the less; which is impossible.

Therefore AG is not parallel to BF.

In the same manner it can be shewn that no other straight line through A but AD is parallel to BF; therefore AD is parallel to BF.

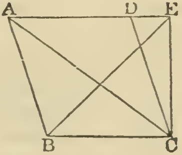
Wherefore, *equal triangles* &c. Q.E.D.

PROPOSITION 41. *THEOREM.*

If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double of the triangle.

Let the parallelogram $ABCD$ and the triangle EBC be on the same base BC, and between the same parallels BC, AE: the parallelogram $ABCD$ shall be double of the triangle EBC.

Join AC.



Then the triangle ABC is equal to the triangle EBC, because they are on the same base BC, and between the same parallels BC, AE. [I. 37.

But the parallelogram $ABCD$ is double of the triangle ABC, because the diameter AC bisects the parallelogram. [I. 34.

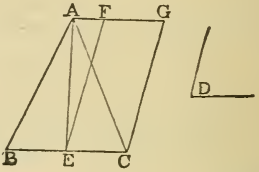
Therefore the parallelogram $ABCD$ is also double of the triangle EBC.

Wherefore, *if a parallelogram* &c. Q.E.D.

PROPOSITION 42. *PROBLEM.*

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle: it is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.



Bisect BC at E: [I. 10.
join AE, and at the point E, in the straight line EC, make the angle CEF equal to D; [I. 23.
through A draw AFG parallel to EC, and through C draw CG parallel to EF. [I. 31.

Therefore $FECG$ is a parallelogram. [*Definition.*

And, because BE is equal to EC, [*Construction.*

the triangle ABE is equal to the triangle AEC, because they are on equal bases BE, EC, and between the same parallels BC, AG. [I. 38.

Therefore the triangle ABC is double of the triangle AEC.

But the parallelogram $FECG$ is also double of the triangle AEC, because they are on the same base EC, and between the same parallels EC, AG. [I. 41.

Therefore the parallelogram $FECG$ is equal to the triangle ABC; [*Axiom* 6.

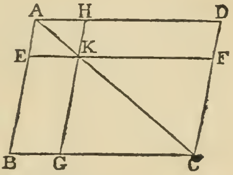
and it has one of its angles CEF equal to the given angle D. [*Construction.*

Wherefore *a parallelogram $FECG$ has been described equal to the given triangle ABC, and having one of its angles CEF equal to the given angle D.* Q.E.F.

PROPOSITION 43. *THEOREM.*

The complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.

Let $ABCD$ be a parallelogram, of which the diameter is AC; and EH, GF parallelograms about AC, that is, through which AC passes; and BK, KD the other parallelograms which make up the whole figure $ABCD$, and which are therefore called the complements: the complement BK shall be equal to the complement KD.



Because $ABCD$ is a parallelogram, and AC its diameter, the triangle ABC is equal to the triangle ADC. [I. 34.

Again, because $AEKH$ is a parallelogram, and AK its diameter, the triangle AEK is equal to the triangle AHK. [I. 34.

For the same reason the triangle KGC is equal to the triangle KFC.

Therefore, because the triangle AEK is equal to the triangle AHK, and the triangle KGC to the triangle KFC; the triangle AEK together with the triangle KGC is equal to the triangle AHK together with the triangle KFC. [*Ax.* 2.

But the whole triangle ABC was shewn to be equal to the whole triangle ADC.

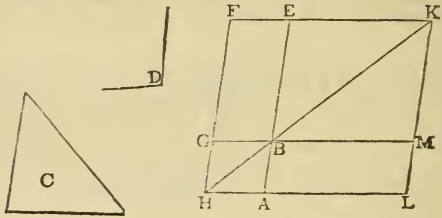
Therefore the remainder, the complement BK, is equal to the remainder, the complement KD. [*Axiom* 3.

Wherefore, *the complements* &c. Q.E.D.

PROPOSITION 44. *PROBLEM.*

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, and C the given triangle, and D the given rectilineal angle: it is required to apply to the straight line AB a parallelogram equal to the triangle C, and having an angle equal to D.



Make the parallelogram $BEFG$ equal to the triangle C, and having the angle EBG equal to the angle D, so that BE may be in the same straight line with AB; [I. 42.
produce FG to H;
through A draw AH parallel to BG or EF, [I. 31.
and join HB.

Then, because the straight line HF falls on the parallels AH, EF, the angles AHF, HFE are together equal to two right angles. [I. 29.
Therefore the angles BHF, HFE are together less than two right angles.
But straight lines which with another straight line make the interior angles on the same side together less than two right angles will meet on that side, if produced far enough. [*Ax.* 12.
Therefore HB and FE will meet if produced;
let them meet at K.
Through K draw KL parallel to EA or FH; [I. 31.
and produce HA, GB to the points L, M.

Then $HLKF$ is a parallelogram, of which the diameter is HK; and AG, ME are parallelograms about HK; and LB, BF are the complements.
Therefore LB is equal to BF. [I. 43.
But BF is equal to the triangle C. [*Construction.*
Therefore LB is equal to the triangle C. [*Axiom* 1.

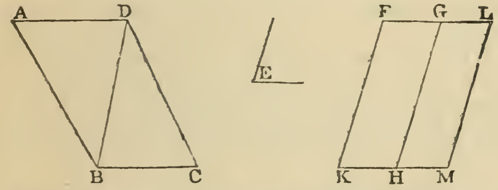
And because the angle GBE is equal to the angle ABM, [I.15.
and likewise to the angle D; [*Construction.*
the angle ABM is equal to the angle D. [*Axiom* 1.

Wherefore *to the given straight line AB the parallelogram LB is applied, equal to the triangle C, and having the angle ABM equal to the angle D.* Q.E.F.

PROPOSITION 45. *PROBLEM.*

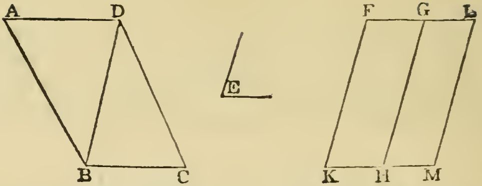
To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let $ABCD$ be the given rectilineal figure, and E the given rectilineal angle: it is required to describe a parallelogram equal to $ABCD$, and having an angle equal to E.



Join DB, and describe the parallelogram FH equal to the triangle ADB, and having the angle FKH equal to the angle E; [I. 42.
and to the straight line GH apply the parallelogram GM equal to the triangle DBC, and having the angle GHM equal to the angle E. [I. 44.
The figure $FKML$ shall be the parallelogram required.

Because the angle E is equal to each of the angles FKH, GHM, [*Construction.*
the angle FKH is equal to the angle GHM. [*Axiom* 1.
Add to each of these equals the angle KHG;
therefore the angles FKH, KHG are equal to the angles KHG, GHM. [*Axiom* 2.
But FKH, KHG are together equal to two right angles; [I.29.
therefore KHG, GHM are together equal to two right angles.



And because at the point H in the straight line GH, the two straight lines KH, HM, on the opposite sides of it, make the adjacent angles together equal to two right angles, KH is in the same straight line with HM. [I. 14.

And because the straight line HG meets the parallels KM, FG, the alternate angles MHG, HGF are equal. [I. 29.
Add to each of these equals the angle HGL;
therefore the angles MHG, HGL, are equal to the angles HGF, HGL. [*Axiom* 2.
But MHG, HGL are together equal to two right angles; [I. 29.
therefore HGF, HGL are together equal to two right angles.
Therefore FG is in the same straight line with GL. [I. 14.

And because KF is parallel to HG, and HG to ML, [*Constr.*
KF is parallel to ML; [I. 30.
and KM, FL are parallels; [*Construction.*
therefore $KFLM$ is a parallelogram. [*Definition.*
And because the triangle ABD is equal to the parallelogram HF, [*Construction.*
and the triangle DBC to the parallelogram GM; [*Constr.*
the whole rectilineal figure $ABCD$ is equal to the whole parallelogram $KFLM$. [*Axiom* 2.

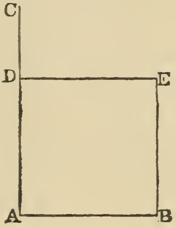
Wherefore, *the parallelogram KFLM has been described equal to the given rectilineal figure ABCD, and having the angle FKM equal to the given angle E.* Q.E.F.

COROLLARY. From this it is manifest, how to a given straight line, to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure: namely, by applying to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle; and so on. [I. 44.

PROPOSITION 46. *PROBLEM.*

To describe a square on a given straight line.

Let AB be the given straight line: it is required to describe a square on AB.



From the point A draw AC at right angles to AB; [I. 11.
and make AD equal to AB; [I. 3.
through D draw DE parallel to AB; and through B draw BE parallel to AD. [I. 31.
$ADEB$ shall be a square.

For $ADEB$ is by construction a parallelogram;
therefore AB is equal to DE, and AD to BE. [I. 34.
But AB is equal to AD. [*Construction.*
Therefore the four straight lines BA, AD, DE, EB are equal to one another, and the parallelogram $ADEB$ is equilateral. [*Axiom* 1.

Likewise all its angles are right angles.

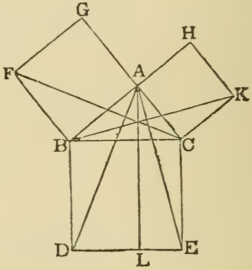
For since the straight line AD meets the parallels AB, DE, the angles BAD, ADE are together equal to two right angles; [I. 29.
but BAD is a right angle; [*Construction.*
therefore also ADE is a right angle. [*Axiom* 3.
But the opposite angles of parallelograms are equal. [I. 34.
Therefore each of the opposite angles ABE, BED is a right angle. [*Axiom* 1.
Therefore the figure $ADEB$ is rectangular;
and it has been shewn to be equilateral.
Therefore *it is a square.* [*Definition* 30.
And it is described on the given straight line AB. Q.E.F.

COROLLARY. From the demonstration it is manifest that every parallelogram which has one right angle has all its angles right angles.

PROPOSITION 47. *THEOREM.*

In any right-angled triangle, the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.

Let ABC be a right-angled triangle, having the right angle BAC: the square described on the side BC shall be equal to the squares described on the sides BA, AC.



On BC describe the square $BDEC$, and on BA, AC describe the squares GB, HC; [I. 46.
through A draw AL parallel to BD or CE; [I. 31.
and join AD, FC.

Then, because the angle BAC is a right angle, [*Hypothesis.*
and that the angle BAG is also a right angle, [*Definition* 30.
the two straight lines AC, AG, on the opposite sides of AB, make with it at the point A the adjacent angles equal to two right angles;
therefore CA is in the same straight line with AG. [I. 14.
For the same reason, AB and AH are in the same straight line.

Now the angle DBC is equal to the angle FBA, for each of them is a right angle. [*Axiom* 11.
Add to each the angle ABC.
Therefore the whole angle DBA is equal to the whole angle FBC. [*Axiom* 2.
And because the two sides AB, BD are equal to the two sides FB, BC, each to each; [*Definition* 30.
and the angle DBA is equal to the angle FBC;
therefore the triangle ABD is equal to the triangle FBC. [I. 4.

Now the parallelogram BL is double of the triangle ABD, because they are on the same base BD, and between the same parallels BD, AL. [I. 41.

And the square GB is double of the triangle FBC, because they are on the same base FB, and between the same parallels FB, GC. [I. 41.

But the doubles of equals are equal to one another. [*Ax.* 6.

Therefore the parallelogram BL is equal to the square GB.

In the same manner, by joining AE, BK, it can be shewn, that the parallelogram CL is equal to the square CH.

Therefore the whole square $BDEC$ is equal to the two squares GB, HC. [*Axiom* 2.

And the square $BDEC$ is described on BC, and the squares GB, HC on BA, AC.

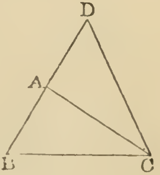
Therefore the square described on the side BC is equal to the squares described on the sides BA, AC.

Wherefore, *in any right-angled triangle* &c. Q.E.D.

PROPOSITION 48. *THEOREM.*

If the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it, the angle contained by these two sides is a right angle.

Let the square described on BC, one of the sides of the triangle ABC, be equal to the squares described on the other sides BA, AC: the angle BAC shall be a right angle.



From the point A draw AD at right angles to AC; [I. 11.

and make AD equal to BA; [I. 3.

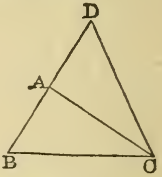
and join DC.

Then because DA is equal to BA, the square on DA is equal to the square on BA.

To each of these add the square on AC.

Therefore the squares on DA, AC are equal to the squares on BA, AC. [*Axiom* 2.

But because the angle DAC is a right angle, [*Construction.*
the square on DC is equal to the squares on DA, AC. [I. 47.
And, by hypothesis, the square on BC is equal to the squares on BA, AC.
Therefore the square on DC is equal to the square on BC. [*Ax.* 1.
Therefore also the side DC is equal to the side BC.



And because the side DA is equal to the side AB; [*Constr.*
and the side AC is common to the two triangles DAC, BAC;
the two sides DA, AC are equal to the two sides BA, AC, each to each;
and the base DC has been shewn to be equal to the base BC;
therefore the angle DAC is equal to the angle BAC. [I. 8.
But DAC is a right angle; [*Construction.*
therefore also BAC is a right angle. [*Axiom* 1.

Wherefore, *if the square* &c. Q.E.D.

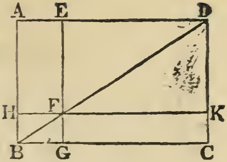
BOOK II.

DEFINITIONS.

1. Every right-angled parallelogram, or rectangle, is said to be contained by any two of the straight lines which contain one of the right angles.

2. In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon.

Thus the parallelogram *HG*, together with the complements *AF*, *FC*, is the gnomon, which is more briefly expressed by the letters *AGK*, or *EHC*, which are at the opposite angles of the parallelograms which make the gnomon.

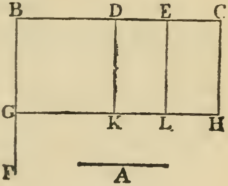


PROPOSITION 1. *THEOREM.*

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let *A* and *BC* be two straight lines; and let *BC* be divided into any number of parts at the points *D*, *E*: the rectangle contained by the straight lines *A*, *BC*, shall be equal to the rectangle contained by *A*, *BD*, together with that contained by *A*, *DE*, and that contained by *A*, *EC*.

From the point *B* draw *BF* at right angles to *BC*; [I. 11.
and make *BG* equal to *A*; [I. 3.
through *G* draw *GH* parallel to *BC*; and through *D*, *E*, *C* draw *DK*, *EL*, *CH*, parallel to *BG*. [I. 31.



Then the rectangle *BH* is equal to the rectangles *BK*, *DL*, *EH*.

But *BH* is contained by *A*, *BC*, for it is contained by *GB*, *BC*, and *GB* is equal to *A*. [*Construction.*

And *BK* is contained by *A*, *BD*, for it is contained by *GB*, *BD*, and *GB* is equal to *A*;

and *DL* is contained by *A*, *DE*, because *DK* is equal to *BG*, which is equal to *A*; [I. 34.

and in like manner *EH* is contained by *A*, *EC*.

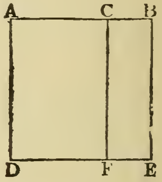
Therefore the rectangle contained by *A*, *BC* is equal to the rectangles contained by *A*, *BD*, and by *A*, *DE*, and by *A*, *EC*.

Wherefore, *if there be two straight lines* &c. Q.E.D.

PROPOSITION 2. *THEOREM.*

If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the square on the whole line.

Let the straight line AB be divided into any two parts at the point C: the rectangle contained by AB, BC, together with the rectangle AB, AC, shall be equal to the square on AB.



[*Note.* To avoid repeating the word *contained* too frequently, the rectangle contained by two straight lines AB, AC is sometimes simply called the rectangle AB, AC.]

On AB describe the square $ADEB$; [I. 46.

and through C draw CF parallel to AD or BE. [I. 31.

Then AE is equal to the rectangles AF, CE.

But AE is the square on AB.

And AF is the rectangle contained by BA, AC, for it is contained by DA, AC, of which DA is equal to BA;

and CE is contained by AB, BC, for BE is equal to AB.

Therefore the rectangle AB, AC, together with the rectangle AB, BC, is equal to the square on AB.

Wherefore, *if a straight line* &c. Q.E.D.

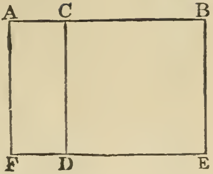
PROPOSITION 3. *THEOREM.*

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square on the aforesaid part.

Let the straight line AB be divided into any two parts at the point C: the rectangle AB, BC shall be equal to the rectangle AC, CB, together with the square on BC.

On BC describe the square $CDEB$; [I. 46.

produce ED to F, and through A draw AF parallel to CD or BE. [I. 31.



Then the rectangle AE is equal to the rectangles AD, CE.

But AE is the rectangle contained by AB, BC, for it is contained by AB, BE, of which BE is equal to BC;

and AD is contained by AC, CB, for CD is equal to CB; and CE is the square on BC.

Therefore the rectangle AB, BC is equal to the rectangle AC, CB, together with the square on BC.

Wherefore, *if a straight line* &c. Q.E.D.

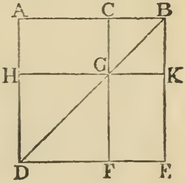
PROPOSITION 4. *THEOREM.*

If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the two parts.

Let the straight line AB be divided into any two parts at the point C: the square on AB shall be equal to the squares on AC, CB, together with twice the rectangle contained by AC, CB.

On AB describe the square $ADEB$; [I. 46.

join BD; through C draw CGF parallel to AD or BE, and through G draw HK parallel to AB or DE. [I. 31.



Then, because CF is parallel to AD, and BD falls on them, the exterior angle CGB is equal to the interior and opposite angle ADB; [I. 29.

but the angle ADB is equal to the angle ABD, [I. 5.

because BA is equal to AD, being sides of a square;

therefore the angle CGB is equal to the angle CBG; [*Ax.* 1.

and therefore the side CG is equal to the side CB. [I. 6.

But CB is also equal to GK, and CG to BK; [I. 34.

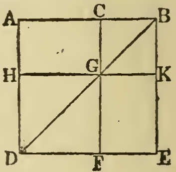
therefore the figure $CGKB$ is equilateral.

It is likewise rectangular. For since CG is parallel to BK, and CB meets them, the angles KBC, GCB are together equal to two right angles. [I. 29.

But KBC is a right angle. [I. *Definition* 30

Therefore GCB is a right angle. [*Axiom* 3.

And therefore also the angles CGK, GKB opposite to these are right angles. [I. 34. and *Axiom* 1.



Therefore $CGKB$ is rectangular; and it has been shewn to be equilateral; therefore it is a square, and it is on the side CB.

For the same reason HF is also a square, and it is on the side HG, which is equal to AC. [I. 34.

Therefore HF, CK are the squares on AC, CB.

And because the complement AG is equal to the complement GE; [I. 43.

and that AG is the rectangle contained by AC, CB, for CG is equal to CB;

therefore GE is also equal to the rectangle AC, CB. [*Ax.* 1.

Therefore AG, GE are equal to twice the rectangle AC, CB.

And HF, CK are the squares on AC, CB.

Therefore the four figures HF, CK, AG, GE are equal to the squares on AC, CB, together with twice the rectangle AC, CB.

But HF, CK, AG, GE make up the whole figure $ADEB$, which is the square on AB.

Therefore the square on AB is equal to the squares on AC, CB, together with twice the rectangle AC, CB.

Wherefore, *if a straight line* &c. Q.E.D.

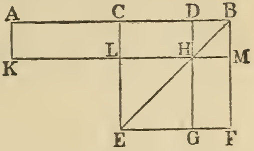
COROLLARY. From the demonstration it is manifest, that parallellograms about the diameter of a square are likewise squares.

PROPOSITION 5. *THEOREM.*

If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the

unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Let the straight line AB be divided into two equal parts at the point C, and into two unequal parts at the point D: the rectangle AD, DB, together with the square on CD, shall be equal to the square on CB.



On CB describe the square $CEFB$; [I. 46.
join BE; through D draw DHG parallel to CE or BF; through H draw KLM parallel to CB or EF; and through A draw AK parallel to CL or BM. [I. 31.

Then the complement CH is equal to the complement HF; [I. 43.

to each of these add DM; therefore the whole CM is equal to the whole DF. [*Axiom* 2.

But CM is equal to AL, [I. 36.

because AC is equal to CB. [*Hypothesis.*

Therefore also AL is equal to DF. [*Axiom* 1.

To each of these add CH; therefore the whole AH is equal to DF and CH. [*Axiom* 2.

But AH is the rectangle contained by AD, DB, for DH is equal to DB; [II. 4, *Corollary.*

and DF together with CH is the gnomon CMG;

therefore the gnomon CMG is equal to the rectangle AD, DB.

To each of these add LG, which is equal to the square on CD. [II. 4, *Corollary*, and I. 34.

Therefore the gnomon CMG, together with LG, is equal to the rectangle AD, DB, together with the square on CD. [*Ax.* 2.

But the gnomon CMG and LG make up the whole figure $CEFB$, which is the square on CB.

Therefore the rectangle AD, DB, together with the square on CD, is equal to the square on CB.

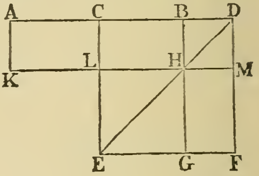
Wherefore, *if a straight line* &c. Q.E.D.

From this proposition it is manifest that the difference of the squares on two unequal straight lines AC, CD, is equal to the rectangle contained by their sum and difference.

PROPOSITION 6. *THEOREM.*

If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.

Let the straight line AB be bisected at the point C, and produced to the point D: the rectangle AD, DB, together with the square on CB, shall be equal to the square on CD.



On CD describe the square $CEFD$; [I. 46.
join DE; through B draw BHG parallel to CE or DF; through H draw KLM parallel to AD or EF; and through A draw AK parallel to CL or DM. [I. 31.

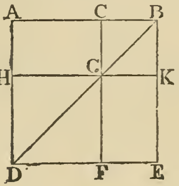
Then, because AC is equal to CB, [*Hypothesis.*
the rectangle AL is equal to the rectangle CH; [I. 36.
but CH is equal to HF; [I. 43.
therefore also AL is equal to HF. [*Axiom* 1.
To each of these add CM;
therefore the whole AM is equal to the gnomon CMG. [*Ax.* 2.
But AM is the rectangle contained by AD, DB, for DM is equal to DB. [II. 4, *Corollary.*
Therefore the rectangle AD, DB is equal to the gnomon CMG. [*Axiom* 1.
To each of these add LG, which is equal to the square on CB. [II. 4, *Corollary*, and I. 34.
Therefore the rectangle AD, DB, together with the square on CB, is equal to the gnomon CMG and the figure LG.
But the gnomon CMG and LG make up the whole figure $CEFD$, which is the square on CD.
Therefore the rectangle AD, DB, together with the square on CB, is equal to the square on CD.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 7. *THEOREM.*

If a straight line be divided into any two parts, the squares on the whole line, and on one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.

Let the straight line AB be divided into any two parts at the point C: the squares on AB, BC shall be equal to twice the rectangle AB, BC, together with the square on AC.



On AB describe the square $ADEB$, and construct the figure as in the preceding propositions.

Then AG is equal to GE; [I. 43.

to each of these add CK;

therefore the whole AK is equal to the whole CE;

therefore AK, CE are double of AK.

But AK, CE are the gnomon AKF, together with the square CK;

therefore the gnomon AKF, together with the square CK, is double of AK.

But twice the rectangle AB, BC is double of AK, for BK is equal to BC. [II. 4, *Corollary.*

Therefore the gnomon AKF, together with the square CK, is equal to twice the rectangle AB, BC.

To each of these equals add HF, which is equal to the square on AC. [II. 4, *Corollary*, and I. 34.

Therefore the gnomon AKF, together with the squares CK, HF, is equal to twice the rectangle AB, BC, together with the square on AC.

But the gnomon AKF together with the squares CK, HF, make up the whole figure $ADEB$ and CK, which are the squares on AB and BC.

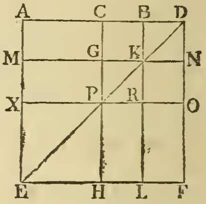
Therefore the squares on AB, BC, are equal to twice the rectangle AB, BC, together with the square on AC.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 8. *THEOREM.*

If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and that part.

Let the straight line AB be divided into any two parts at the point C: four times the rectangle AB, BC, together with the square on AC, shall be equal to the square on the straight line made up of AB and BC together.



Produce AB to D, so that BD may be equal to CB; [*Post.* 2. and I. 3.

on AD describe the square $AEFD$;

and construct two figures such as in the preceding propositions.

Then, because CB is equal to BD, [*Construction.*

and that CB is equal to GK, and BD to KN, [I. 34.

therefore GK is equal to KN. [*Axiom* 1.

For the same reason PR is equal to RO.

And because CB is equal to BD, and GK to KN, the rectangle CK is equal to the rectangle BN, and the rectangle GR to the rectangle RN. [I. 36.

But CK is equal to RN, because they are the complements of the parallelogram CO; [I. 43.

therefore also BN is equal to GR. [*Axiom* 1.

Therefore the four rectangles BN, CK, GR, RN are equal to one another, and so the four are quadruple of one of them CK.

Again, because CB is equal to BD, [*Construction.*

and that BD is equal to BK, [II. 4, *Corollary.*

that is to CG, [I. 34.

and that CB is equal to GK, [I. 34.

that is to GP; [II. 4, *Corollary.*
therefore CG is equal to GP. [*Axiom* 1.

And because CG is equal to GP, and PR to RO, the rectangle AG is equal to the rectangle MP, and the rectangle PL to the rectangle RF. [I. 36.

But MP is equal to PL, because they are the complements of the parallelogram ML; [I. 43.

therefore also AG is equal to RF. [*Axiom* 1.

Therefore the four rectangles AG, MP, PL, RF are equal to one another, and so the four are quadruple of one of them AG.

And it was shewn that the four CK, BN, GR and RN are quadruple of CK; therefore the eight rectangles which make up the gnomon AOH are quadruple of AK.

And because AK is the rectangle contained by AB, BC, for BK is equal to BC;

therefore four times the rectangle AB, BC is quadruple of AK.

But the gnomon AOH was shewn to be quadruple of AK.

Therefore four times the rectangle AB, BC is equal to the gnomon AOH. [*Axiom* 1.

To each of these add XH, which is equal to the square on AC. [II. 4, *Corollary*, and I. 34.

Therefore four times the rectangle AB, BC, together with the square on AC, is equal to the gnomon AOH and the square XH.

But the gnomon AOH and the square XH make up the figure $AEFD$, which is the square on AD.

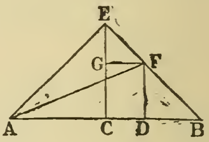
Therefore four times the rectangle AB, BC, together with the square on AC, is equal to the square on AD, that is to the square on the line made of AB and BC together.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 9. *THEOREM.*

If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.

Let the straight line AB be divided into two equal parts at the point C, and into two unequal parts at the point D: the squares on AD, DB shall be together double of the squares on AC, CD.



From the point C draw CE at right angles to AB, [I. 11.
and make it equal to AC or CB, [I. 3.
and join EA, EB; through D draw DF parallel to CE, and through F draw FG parallel to BA; [I. 31.
and join AF.

Then, because AC is equal to CE, [*Construction.*
the angle EAC is equal to the angle AEC. [I. 5.
And because the angle ACE is a right angle, [*Construction.*
the two other angles AEC, EAC are together equal to one right angle; [I. 32.
and they are equal to one another;
therefore each of them is half a right angle.
For the same reason each of the angles CEB, EBC is half a right angle.
Therefore the whole angle AEB is a right angle.

And because the angle GEF is half a right angle, and the angle EGF a right angle, for it is equal to the interior and opposite angle ECB; [I. 29.
therefore the remaining angle EFG is half a right angle.
Therefore the angle GEF is equal to the angle EFG, and the side EG is equal to the side GF. [I. 6.
Again, because the angle at B is half a right angle, and the

angle FDB a right angle, for it is equal to the interior and opposite angle ECB; [I. 29.

therefore the remaining angle BFD is half a right angle. Therefore the angle at B is equal to the angle BFD, and the side DF is equal to the side DB. [I. 6.

And because AC is equal to CE, [*Construction.*

the square on AC is equal to the square on CE;

therefore the squares on AC, CE are double of the square on AC.

But the square on AE is equal to the squares on AC, CE, because the angle ACE is a right angle; [I. 47.

therefore the square on AE is double of the square on AC.

Again, because EG is equal to GF, [*Construction.*

the square on EG is equal to the square on GF;

therefore the squares on EG, GF are double of the square on GF.

But the square on EF is equal to the squares on EG, GF, because the angle EGF is a right angle; [I. 47.

therefore the square on EF is double of the square on GF.

And GF is equal to CD; [I. 34.

therefore the square on EF is double of the square on CD.

But it has been shewn that the square on AE is also double of the square on AC.

Therefore the squares on AE, EF are double of the squares on AC, CD.

But the square on AF is equal to the squares on AE, EF, because the angle AEF is a right angle. [I. 47.

Therefore the square on AF is double of the squares on AC, CD.

But the squares on AD, DF are equal to the square on AF, because the angle ADF is a right angle. [I. 47.

Therefore the squares on AD, DF are double of the squares on AC, CD.

And DF is equal to DB;

therefore the squares on AD, DB are double of the squares on AC, CD.

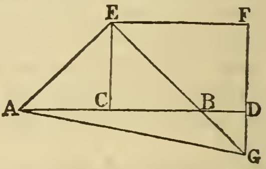
Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 10. *THEOREM.*

If a straight line be bisected, and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line bisected and of the square on the line made up of the half and the part produced.

Let the straight line *AB* be bisected at *C*, and produced to *D*: the squares on *AD*, *DB* shall be together double of the squares on *AC*, *CD*.

From the point *C* draw *CE* at right angles to *AB*, [I. 11.
and make it equal to *AC* or *CB*; [I. 3.
and join *AE*, *EB*; through *E* draw *EF* parallel to *AB*, and through *D* draw *DF* parallel to *CE*. [I. 31.



Then because the straight line *EF* meets the parallels *EC*, *FD*, the angles *CEF*, *EFD* are together equal to two right angles; [I. 29.
and therefore the angles *BEF*, *EFD* are together less than two right angles.
Therefore the straight lines *EB*, *FD* will meet, if produced, towards *B*, *D*. [*Axiom* 12.
Let them meet at *G*, and join *AG*.

Then because *AC* is equal to *CE*, [*Construction.*
the angle *CEA* is equal to the angle *EAC*; [I. 5.
and the angle *ACE* is a right angle; [*Construction.*
therefore each of the angles *CEA*, *EAC* is half a right angle. [I. 32.
For the same reason each of the angles *CEB*, *EBC* is half a right angle.
Therefore the angle *AEB* is a right angle.

And because the angle *EBC* is half a right angle, the angle *DBG* is also half a right angle, for they are vertically opposite; [I. 15.
but the angle *BDG* is a right angle, because it is equal to the alternate angle *DCE*; [I. 29.
therefore the remaining angle *DGB* is half a right angle, [I. 32.

and is therefore equal to the angle DBG;

therefore also the side BD is equal to the side DG. [I. 6.

Again, because the angle EGF is half a right angle, and the angle at F a right angle, for it is equal to the opposite angle ECD; [I. 34.

therefore the remaining angle FEG is half a right angle, [I. 32.

and is therefore equal to the angle EGF;

therefore also the side GF is equal to the side FE. [I. 6.

And because EC is equal to CA, the square on EC is equal to the square on CA;

therefore the squares on EC, CA are double of the square on CA.

But the square on AE is equal to the squares on EC, CA. [I. 47.

Therefore the square on AE is double of the square on AC.

Again, because GF is equal to FE, the square on GF is equal to the square on FE;

therefore the squares on GF, FE are double of the square on FE.

But the square on EG is equal to the squares on GF, FE. [I. 47.

Therefore the square on EG is double of the square on FE.

And FE is equal to CD; [I. 34.

therefore the square on EG is double of the square on CD.

But it has been shewn that the square on AE is double of the square on AC.

Therefore the squares on AE, EG are double of the squares on AC, CD.

But the square on AG is equal to the squares on AE, EG. [I. 47.

Therefore the square on AG is double of the squares on AC, CD.

But the squares on AD, DG are equal to the square on AG. [I. 47.

Therefore the squares on AD, DG are double of the squares on AC, CD.

And DG is equal to DB;

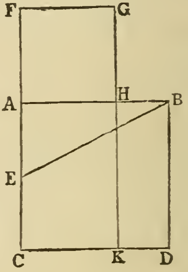
therefore the squares on AD, DB are double of the squares on AC, CD.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 11. *PROBLEM.*

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.

Let AB be the given straight line: it is required to divide it into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.



On AB describe the square $ABDC$; [I. 46.

bisect AC at E; [I. 10.

join BE; produce CA to F, and make EF equal to EB; [I. 3.

and on AF describe the square $AFGH$. [I. 46.

AB shall be divided at H so that the rectangle AB, BH is equal to the square on AH.

Produce GH to K.

Then, because the straight line AC is bisected at E, and produced to F, the rectangle CF, FA, together with the square on AE, is equal to the square on EF. [II. 6.

But EF is equal to EB. [*Construction.*

Therefore the rectangle CF, FA, together with the square on AE, is equal to the square on EB.

But the square on EB is equal to the squares on AE, AB, because the angle EAB is a right angle. [I. 47.

Therefore the rectangle CF, FA, together with the square on AE, is equal to the squares on AE, AB.

Take away the square on AE, which is common to both; therefore the remainder, the rectangle CF, FA, is equal to the square on AB. [*Axiom* 3.

But the figure FK is the rectangle contained by CF, FA, for FG is equal to FA;

and AD is the square on AB;

therefore FK is equal to AD.

Take away the common part AK, and the remainder FH is equal to the remainder HD. [*Axiom* 3.

But HD is the rectangle contained by AB, BH, for AB is equal to BD;

and FH is the square on AH;

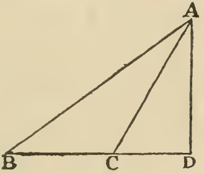
therefore the rectangle AB,BH is equal to the square on AH.

Wherefore *the straight line AB is divided at H, so that the rectangle AB, BH is equal to the square on AH.* Q.E.F.

PROPOSITION 12. *THEOREM.*

In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the straight line intercepted without the triangle, between the perpendicular and the obtuse angle.

Let ABC be an obtuse-angled triangle, having the obtuse angle ACB, and from the point A let AD be drawn perpendicular to BC produced: the square on AB shall be greater than the squares on AC, CB, by twice the rectangle BC, CD.



Because the straight line BD is divided into two parts at the point C, the square on BD is equal to the squares on BC, CD, and twice the rectangle BC, CD. [II. 4.

To each of these equals add the square on DA.

Therefore the squares on BD, DA are equal to the squares on BC, CD, DA, and twice the rectangle BC, CD. [*Axiom* 2.

But the square on BA is equal to the squares on BD, DA, because the angle at D is a right angle; [I. 47.

and the square on CA is equal to the squares on CD,DA. [I. 47.

Therefore the square on BA is equal to the squares on BC, CA, and twice the rectangle BC, CD;

that is, the square on BA is greater than the squares on BC, CA by twice the rectangle BC, CD.

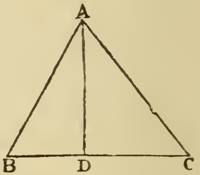
Wherefore, *in obtuse-angled triangles* &c. Q.E.D.

PROPOSITION 13. *THEOREM.*

In every triangle, the square on the side subtending an acute angle, is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle, and the acute angle.

Let ABC be any triangle, and the angle at B an acute angle; and on BC one of the sides containing it, let fall the perpendicular AD from the opposite angle: the square on AC, opposite to the angle B, shall be less than the squares on CB, BA, by twice the rectangle CB, BD.

First, let AD fall within the triangle ABC.



Then, because the straight line CB is divided into two parts at the point D, the squares on CB, BD are equal to twice the rectangle contained by CB, BD and the square on CD. [II. 7.

To each of these equals add the square on DA.

Therefore the squares on CB, BD, DA are equal to twice the rectangle CB, BD and the squares on CD, DA. [*Ax.* 2.

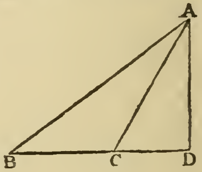
But the square on AB is equal to the squares on BD, DA, because the angle BDA is a right angle; [I. 47.

and the square on AC is equal to the squares on CD, DA. [I. 47.

Therefore the squares on CB, BA are equal to the square on AC and twice the rectangle CB, BD;

that is, the square on AC alone is less than the squares on CB, BA by twice the rectangle CB, BD.

Secondly, let AD fall without the triangle ABC.



Then because the angle at D is a right angle, [*Construction.*

the angle ACB is greater than a right angle; [I. 16.

and therefore the square on AB is equal to the squares on AC, CB, and twice the rectangle BC, CD. [II. 12.

To each of these equals add the square on BC.

Therefore the squares on AB, BC are equal to the square on AC, and twice the square on BC, and twice the rectangle BC, CD. [*Axiom* 2.

But because BD is divided into two parts at C, the rectangle DB, BC is equal to the rectangle BC, CD and the square on BC; [II. 3.

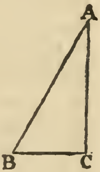
and the doubles of these are equal,

that is, twice the rectangle DB, BC is equal to twice the rectangle BC, CD and twice the square on BC.

Therefore the squares on AB, BC are equal to the square on AC, and twice the rectangle DB, BC;

that is, the square on AC alone is less than the squares on AB, BC by twice the rectangle DB, BC.

Lastly, let the side AC be perpendicular to BC.



Then BC is the straight line between the perpendicular and the acute angle at B;

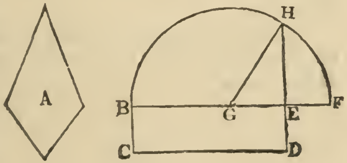
and it is manifest, that the squares on AB, BC are equal to the square on AC, and twice the square on BC. [I. 47 and *Ax.* 2.

Wherefore, *in every triangle* &c. Q.E.D.

PROPOSITION 14. *PROBLEM.*

To describe a square that shall be equal to a given rectilineal figure.

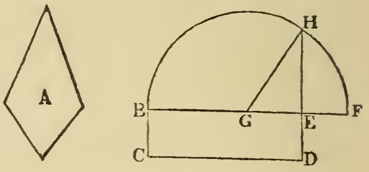
Let A be the given rectilineal figure: it is required to describe a square that shall be equal to A.



Describe the rectangular parallelogram $BCDE$ equal to the rectilineal figure A. [I. 45.

Then if the sides of it, BE, ED are equal to one another, it is a square, and what was required is now done.

But if they are not equal, produce one of them BE to F, make EF equal to ED, [I. 3. and bisect BF at G; [I. 10. from the centre G, at the distance GB, or GF, describe the semicircle BHF, and produce DE to H.



The square described on EH shall be equal to the given rectilineal figure A.

Join GH. Then, because the straight line BF is divided into two equal parts at the point G, and into two unequal parts at the point E, the rectangle BE, EF, together with the square on GE, is equal to the square on GF. [II. 5.

But GF is equal to GH.

Therefore the rectangle BE, EF, together with the square on GE, is equal to the square on GH.

But the square on GH is equal to the squares on GE, EH; [I. 47.

therefore the rectangle BE, EF, together with the square on GE, is equal to the squares on GE, EH.

Take away the square on GE, which is common to both;

therefore the rectangle BE, EF is equal to the square on EH. [*Axiom* 3.

But the rectangle contained by BE, EF is the parallelogram BD,

because EF is equal to ED. [*Construction.*

Therefore BD is equal to the square on EH.

But BD is equal to the rectilineal figure A. [*Construction.*

Therefore the square on EH is equal to the rectilineal figure A.

Wherefore *a square has been made equal to the given rectilineal figure* A, *namely, the square described on* EH. Q.E.F.

NOTES ON EUCLID'S ELEMENTS.

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THE article Eucleides in Dr Smith's *Dictionary of Greek and Roman Biography* was written by Professor De Morgan; it contains an account of the works of Euclid, and of the various editions of them which have been published. To that article we refer the student who desires full information on these subjects. Perhaps the only work of importance relating to Euclid which has been published since the date of that article is a work on the Porisms of Euclid by Chasles; Paris, 1860.

Euclid appears to have lived in the time of the first Ptolemy, B.C. 323—283, and to have been the founder of the Alexandrian mathematical school. The work on Geometry known as *The Elements of Euclid* consists of thirteen books; two other books have sometimes been added, of which it is supposed that Hypsicles was the author. Besides the *Elements*, Euclid was the author of other works, some of which have been preserved and some lost.

We will now mention the three editions which are the most valuable for those who wish to read the Elements of Euclid in the original Greek.

(1) The Oxford edition in folio, published in 1703 by David Gregory, under the title Εὐκλείδου τὰ σωζόμενα. "As an edition of the whole of Euclid's works, this stands alone, there being no other in Greek." *De Morgan.*

(2) *Euclidis Elementorum Libri sex priores...edidit Joannes Gulielmus Camerer.* This edition was published at Berlin in two volumes octavo, the first volume in 1824 and the second in 1825. It contains the first six books of the *Elements* in Greek with a Latin Translation, and very good notes which form a mathematical commentary on the subject.

(3) *Euclidis Elementa ex optimis libris in usum tironum Græce edita ab Ernesto Ferdinando August.* This edition was published at Berlin in two volumes octavo, the first volume in 1826 and the second in 1829. It contains the thirteen books of the *Elements* in Greek, with a collection of various readings.

A third volume, which was to have contained the remaining works of Euclid, never appeared. "To the scholar who wants one edition of the Elements we should decidedly recommend this, as bringing together all that has been done for the text of Euclid's greatest work." *De Morgan.*

An edition of the whole of Euclid's works in the original has long been promised by Teubner the well-known German publisher, as one of his series of compact editions of Greek and Latin authors; but we believe there is no hope of its early appearance.

Robert Simson's edition of the *Elements of Euclid*, which we have in substance adopted in the present work, differs considerably from the original. The English reader may ascertain the contents of the original by consulting the work entitled *The Elements of Euclid with dissertations...by James Williamson.* This work consists of two volumes quarto; the first volume was published at Oxford in 1781, and the second at London in 1788. Williamson gives a close translation of the thirteen books of the *Elements* into English, and he indicates by the use of Italics the words which are not in the original but which are required by our language.

Among the numerous works which contain notes on the *Elements of Euclid* we will mention four by which we have been aided in drawing up the selection given in this volume.

An Examination of the first six Books of Euclid's Elements by William Austin...Oxford, 1781.

Euclid's Elements of Plane Geometry with copious notes...by John Walker. London, 1827.

The first six books of the Elements of Euclid with a Commentary...by Dionysius Lardner, fourth edition. London, 1834.

Short supplementary remarks on the first six Books of Euclid's Elements, by Professor De Morgan, in the *Companion to the Almanac* for 1849.

We may also notice the following works:

Geometry, Plane, Solid, and Spherical,...London 1830; this forms part of the Library of Useful Knowledge.

Théorèmes et Problèmes de Géométrie Elementaire par Eugène Catalan.. Troisième édition. Paris, 1858.

For the History of Geometry the student is referred to Montucla's *Histoire des Mathématiques*, and to Chasles's *Aperçu historique sur l'origine et le devéloppement des Méthodes en Géométrie...*

THE FIRST BOOK.

Definitions. The first seven definitions have given rise to considerable discussion, on which however we do not propose to enter. Such a discussion would consist mainly of two subjects, both of which are unsuitable to an elementary work, namely, an examination of the origin and nature of some of our elementary ideas, and a comparison of the original text of Euclid with the substitutions for it proposed by Simson and other editors. For the former subject the student may hereafter consult Whewell's *History of Scientific Ideas* and Mill's *Logic*, and for the latter the notes in Camerer's edition of the *Elements of Euclid.*

We will only observe that the ideas which correspond to the words *point, line*, and *surface*, do not admit of such definitions as will really supply the ideas to a person who is destitute of them. The so-called definitions may be regarded as cautions or restrictions. Thus a *point* is not to be supposed to have any *size*, but only *position;* a line is not to be supposed to have any *breadth* or *thickness*, but only length; a surface is not to be supposed to have any *thickness*, but only *length* and *breadth.*

The eighth definition seems intended to include the cases in which an angle is formed by the meeting of two *curved* lines, or of a *straight* line and a *curved* line; this definition however is of no importance, as the only angles ever considered are such as are formed by straight lines. The definition of a plane rectilineal angle is important; the beginner must carefully observe that no change is made in an angle by prolonging the lines which form it, away from the angular point.

Some writers object to such definitions as those of an equilateral triangle, or of a square, in which the existence of the object defined is *assumed* when it ought to be *demonstrated.* They would present them in such a form as the following: if there be a triangle having three equal sides, let it be called an equilateral triangle.

Moreover, some of the definitions are introduced prematurely. Thus, for example, take the definitions of a right-angled triangle and an obtuse-angled triangle; it is not shewn until I. 17, that a triangle cannot have both a right angle and an obtuse angle, and so cannot be at the same time right-angled and obtuse-angled. And before Axiom 11 has been given, it is conceivable

that the same angle may be greater than one right angle, and less than another right angle, that is, obtuse and acute at the same time.

The definition of a square assumes more than is necessary. For if a four-sided figure have all its sides equal and *one* angle a right angle, it may be shewn that *all* its angles are right angles; or if a four-sided figure have all its angles *equal*, it may be shewn that they are all *right angles*.

Postulates. The postulates state what processes we assume that we can effect, namely, that we can draw a straight line between two given points, that we can produce a straight line to any length, and that we can describe a circle from a given centre with a given distance as radius. It is sometimes stated that the postulates amount to requiring the use of a *ruler* and *compasses*. It must however be observed that the ruler is not supposed to be a *graduated* ruler, so that we cannot use it to measure off assigned lengths. And we do not require the compasses for any other process than describing a circle from a given point with a given distance as radius; in other words, the compasses may be supposed to close of themselves, as soon as one of their points is removed from the paper.

Axioms. The axioms are called in the original *Common Notions*. It is supposed by some writers that Euclid intended his postulates to include all demands which are peculiarly geometrical, and his common notions to include only such notions as are applicable to all kinds of magnitude as well as to space magnitudes. Accordingly, these writers remove the last three axioms from their place and put them among the postulates; and this transposition is supported by some manuscripts and some versions of the *Elements*.

The fourth axiom is sometimes referred to in editions of Euclid when in reality more is required than this axiom expresses. Euclid says, that if A and B be unequal, and C and D equal, the sum of A and C is *unequal* to the sum of B and D. What Euclid often requires is something more, namely, that if A be greater than B, and C and D be equal, the sum of A and C is *greater* than the sum of B and D. Such an axiom as this is required, for example, in I. 17. A similar remark applies to the fifth axiom.

In the eighth axiom the words "that is, which exactly fill the same space," have been introduced without the authority of

the original Greek. They are objectionable, because *lines* and *angles* are magnitudes to which the axiom may be applied, but they cannot be said to *fill space*.

On the *method of superposition* we may refer to papers by Professor Kelland in the *Transactions of the Royal Society of Edinburgh*, Vols. XXI. and XXIII.

The eleventh axiom is not required before I. 14, and the twelfth axiom is not required before I. 29; we shall not consider these axioms until we arrive at the propositions in which they are respectively required for the first time.

The first book is chiefly devoted to the properties of triangles and parallelograms.

We may observe that Euclid himself does not distinguish between problems and theorems except by using at the end of the investigation phrases which correspond to Q.E.F. and Q.E.D. respectively.

I. 2. This problem admits of *eight* cases in its figure. For it will be found that the given point may be joined with *either* end of the given straight line, then the equilateral triangle may be described on *either* side of the straight line which is drawn, and the sides of the equilateral triangle which are produced may be produced through *either* extremity. These various cases may be left for the exercise of the student, as they present no difficulty.

There will not however always be eight different straight lines obtained which solve the problem. For example, if the point A falls on BC produced, some of the solutions obtained coincide; this depends on the fact which follows from I. 32, that the angles of all equilateral triangles are equal.

I. 5. "Join FC." Custom seems to allow this singular expression as an abbreviation for "draw the straight line FC," or for "join F to C by the straight line FC."

In comparing the triangles BFC, CGB, the words "and the base BC is common to the two triangles BFC, CGB" are usually inserted, with the authority of the original. As however these words are of no use, and tend to perplex a beginner, we have followed the example of some editors and omitted them.

A *corollary* to a proposition is an inference which may be deduced immediately from that proposition. Many of the corollaries in the *Elements* are not in the original text, but introduced by the editors.

It has been suggested to demonstrate I. 5 by *superposition*. Conceive the isosceles triangle ABC to be taken up, and then replaced so that AB falls on the old position of AC, and AC falls on the old position of AB. Thus, in the manner of I. 4, we can shew that the angle ABC is equal to the angle ACB.

I. 6 is the *converse* of part of I. 5. One proposition is said to be the converse of another when the conclusion of each is the hypothesis of the other. Thus in I. 5 the hypothesis is the equality of the sides, and one conclusion is the equality of the angles; in I. 6 the hypothesis is the equality of the angles and the conclusion is the equality of the sides. When there is more than one hypothesis or more than one conclusion to a proposition, we can form more than one converse proposition. For example, as another converse of I. 5 we have the following: if the angles formed by the base of a triangle and the sides produced be equal, the sides of the triangle are equal; this proposition is true and will serve as an exercise for the student.

The converse of a true proposition is not necessarily true; the student however will see, as he proceeds, that Euclid shews that the converses of many geometrical propositions are true.

I. 6 is an example of the *indirect* mode of demonstration, in which a result is established by shewing that some absurdity follows from supposing the required result to be untrue. Hence this mode of demonstration is called the *reductio ad absurdum*. Indirect demonstrations are often less esteemed than direct demonstrations; they are said to shew that a theorem *is* true rather than to shew *why* it is true. Euclid uses the *reductio ad absurdum* chiefly when he is demonstrating the converse of some former theorem; see I. 14, 19, 25, 40.

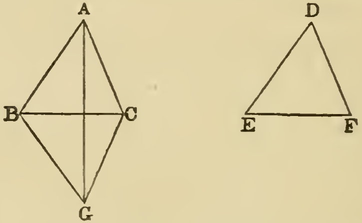
Some remarks on *indirect demonstration* by Professor Sylvester, Professor De Morgan, and Dr Adamson will be found in the volumes of the *Philosophical Magazine* for 1852 and 1853.

I. 6 is not required by Euclid before he reaches II. 4; so that I. 6 might be removed from its present place and demonstrated hereafter in other ways if we please. For example, I. 6 might be placed after I. 18 and demonstrated thus. Let the angle ABC be equal to the angle ACB: then the side AB shall be equal to the side AC. For if not, one of them must be greater than the other; suppose AB greater than AC. Then the angle ACB is greater than the angle ABC, by I. 18. But this is impossible, because

the angle ACB is equal to the angle ABC, by hypothesis. Or I. 6 might be placed after I. 26 and demonstrated thus. Bisect the angle BAC by a straight line meeting the base at D. Then the triangles ABD and ACD are equal in all respects, by I. 26.

I. 7 is only required in order to lead to I. 8. The two might be superseded by another demonstration of I. 8, which has been recommended by many writers.

Let ABC, DEF be two triangles, having the sides AB, AC equal to the sides DE, DF, each to each, and the base BC equal to the base EF: the angle BAC shall be equal to the angle EDF.



For, let the triangle DEF be applied to the triangle ABC, so that the bases may coincide, the equal sides be conterminous, and the vertices fall on opposite sides of the base. Let GBC represent the triangle DEF thus applied, so that G corresponds to D. Join AG. Since, by hypothesis, BA is equal to BG, the angle BAG is equal to the angle BGA, by I. 5. In the same manner the angle CAG is equal to the angle CGA. Therefore the whole angle BAC is equal to the whole angle BGC, that is, to the angle EDF.

There are two other cases; for the straight line AG may pass through B or C, or it may fall outside BC: these cases may be treated in the same manner as that which we have considered.

I. 8. It may be observed that the two triangles in I. 8 are equal in *all respects;* Euclid however does not assert more than the equality of the angles opposite to the bases, and when he requires more than this result he obtains it by using I. 4.

I. 9. Here the equilateral triangle DEF is to be described on the side *remote* from A, because if it were described on the *same* side, its vertex, F, might coincide with A, and then the construction would fail.

I. 11. The corollary was added by Simson. It is liable to serious objection. For we do not know how the perpendicular BE is to be drawn. If we are to use I. 11 we must produce AB, and then we must assume that there is only *one* way of producing AB, for otherwise we shall not know that there is only *one* perpendicular; and thus we assume what we have to demonstrate.

Simson's corollary might come after I. 13 and be demonstrated thus. If possible let the two straight lines ABC, ABD have the segment AB common to both. From the point B draw any straight line BE. Then the angles ABE and EBC are equal to two right angles, by I. 13, and the angles ABE and EBD are also equal to two right angles, by I. 13. Therefore the angles ABE and EBC are equal to the angles ABE and EBD. Therefore the angle EBC is equal to the angle EBD; which is absurd.

But if the question whether two straight lines can have a common segment is to be considered at all in the Elements, it might occur at an earlier place than Simson has assigned to it. For example, in the figure to I. 5, if two straight lines could have a common segment AB, and then separate at B, we should obtain two different angles formed on the other side of BC by these produced parts, and each of them would be equal to the angle BCG. The opinion has been maintained that even in I. 1, it is tacitly assumed that the straight lines AC and BC cannot have a common segment at C where they meet; see Camerer's *Euclid*, pages 30 and 36.

Simson never formally refers to his corollary until XI. 1. The corollary should be omitted, and the tenth axiom should be extended so as to amount to the following; if two straight lines coincide in two points they must coincide both beyond and between those points.

I. 12. Here the straight line is said to be of *unlimited* length, in order that we may ensure that it shall meet the circle.

Euclid distinguishes between the terms *at right angles* and *perpendicular*. He uses the term *at right angles* when the straight line is drawn from a point *in* another, as in I. 11; and he uses the term *perpendicular* when the straight line is drawn from a point *without* another, as in I. 12. This distinction however is often disregarded by modern writers.

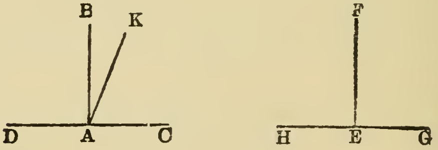
I. 14. Here Euclid first requires his eleventh axiom. For

in the demonstration we have the angles ABC and ABE equal to two right angles, and also the angles ABC and ABD equal to two right angles; and then the former two right angles are equal to the latter two right angles by the aid of the eleventh axiom. Many modern editions of Euclid however refer *only* to the first axiom, as if that alone were sufficient; a similar remark applies to the demonstrations of I. 15, and I. 24. In these cases we have omitted the reference purposely, in order to avoid perplexing a beginner; but when his attention is thus drawn to the circumstance he will see that the first and eleventh axioms are both used.

We may observe that errors, in the references with respect to the eleventh axiom, occur in other places in many modern editions of Euclid. Thus for example in III. 1, at the step "therefore the angle FDB is equal to the angle GDB," a reference is given to the first axiom *instead* of to the eleventh.

There seems no objection on Euclid's principles to the following *demonstration* of his eleventh axiom.

Let AB be at right angles to DAC at the point A, and EF at right angles to HEG at the point E: then shall the angles BAC and FEG be equal.



Take any length AC, and make AD, EH, EG all equal to AC. Now apply HEG to DAC, so that H may be on D, and HG on DC, and B and F on the same side of DC; then G will coincide with C, and E with A. Also EF shall coincide with AB; for if not, suppose, if possible, that it takes a different position as AK. Then the angle DAK is equal to the angle HEF, and the angle CAK to the angle GEF; but the angles HEF and GEF are equal, by hypothesis; therefore the angles DAK and CAK are equal. But the angles DAB and CAB are also equal, by hypothesis; and the angle CAB is greater than the angle CAK; there-

fore the angle DAB is greater than the angle CAK. Much more then is the angle DAK greater than the angle CAK. But the angle DAK was shewn to be equal to the angle CAK; which is absurd. Therefore EF must coincide with AB; and therefore the angle FEG coincides with the angle BAC, and is equal to it.

I. 18, I. 19. In order to assist the student in remembering which of these two propositions is demonstrated directly and which indirectly, it may be observed that the order is similar to that in I. 5 and I. 6.

I. 20. "Proclus, in his commentary, relates, that the Epicureans derided Prop. 20, as being manifest even to asses, and needing no demonstration; and his answer is, that though the truth of it be manifest to our senses, yet it is science which must give the reason why two sides of a triangle are greater than the third: but the right answer to this objection against this and the 21st, and some other plain propositions, is, that the number of axioms ought not to be increased without necessity, as it must be if these propositions be not demonstrated." *Simson.*

I. 21. Here it must be carefully observed that the two straight lines are to be drawn *from the ends of the side* of the triangle. If this condition be omitted the two straight lines will not necessarily be less than two sides of the triangle.

I. 22. "Some authors blame Euclid because he does not demonstrate that the two circles made use of in the oonstruction of this problem must cut one another: but this is very plain from the determination he has given, namely, that any two of the straight lines DF, FG, GH, must be greater than the third. For who is so dull, though only beginning to learn the Elements, as not to perceive that the circle described from the centre F, at the distance FD, must meet FH betwixt F and H, because FD is less than FH; and that for the like reason, the circle described from the centre G, at the distance GH...must meet DG betwixt D and G; and that these circles must meet one another, because FD and GH are together greater than FG?" *Simson.*

The condition that B and C are greater than A, ensures that the circle described from the centre G shall not fall entirely within the circle described from the centre F; the condition that A and B are greater than C, ensures that the circle described

from the centre F shall not fall entirely within the circle described from the centre G; the condition that A and C are greater than B, ensures that one of these circles shall not fall entirely without the other. Hence the circles must meet. It is easy to *see* this as Simson says, but there is something arbitrary in Euclid's selection of what is to be *demonstrated* and what is to be *seen*, and Simson's language suggests that he was really conscious of this.

I. 24. In the construction, the condition that DE is to be the side which is not greater than the other, was added by Simson; unless this condition be added there will be *three* cases to consider, for F may fall *on* EG, or *above* EG, or *below* EG. It may be objected that even if Simson's condition be added, it ought to be shewn that F will fall *below* EG. Simson accordingly says "...it is very easy to perceive, that DG being equal to DF, the point G is in the circumference of a circle described from the centre D at the distance DF, and must be in that part of it which is above the straight line EF, because DG falls above DF, the angle EDG being greater than the angle EDF." Or we may shew it in the following manner. Let H denote the point of intersection of DF and EG. Then, the angle DHG is greater than the angle DEG, by I. 16; the angle DEG is not less than the angle DGE, by I. 19; therefore the angle DHG is greater than the angle DGH. Therefore DH is less than DG, by I. 20. Therefore DH is less than DF.

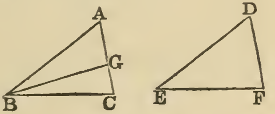
If Simson's condition be omitted, we shall have two other cases to consider besides that in Euclid. If F falls *on* EG, it is obvious that EF is less than EG. If F falls *above* EG, the sum of DF and EF is less than the sum of DG and EG, by I. 21; and therefore EF is less than EG.

I. 26. It will appear after I. 32 that two triangles which have two angles of the one equal to two angles of the other, each to each, have also their third angles equal. Hence we are able to include the two cases of I. 26 in one enunciation thus; *if two triangles have all the angles of the one respectively equal to all the angles of the other, each to each, and have also a side of the one, opposite to any angle, equal to the side opposite to the equal angle in the other, the triangles shall be equal in all respects.*

The first twenty-six propositions constitute a distinct section

of the first Book of the *Elements*. The principal results are those contained in Propositions 4, 8, and 26; in each of these Propositions it is shewn that two triangles which agree in three respects agree entirely. There are two other cases which will naturally occur to a student to consider besides those in Euclid; namely, (1) when two triangles have the three angles of the one respectively equal to the three angles of the other, (2) when two triangles have two sides of the one equal to two sides of the other, each to each, and an angle opposite to one side of one triangle equal to the angle opposite to the equal side of the other triangle. In the first of these two cases the student will easily see, after reading I. 29, that the two triangles are not necessarily equal. In the second case also the triangles are not necessarily equal, as may be shewn by an example; in the figure of I. 11, suppose the straight line *FB* drawn; then in the two triangles *FBE*, *FBD*, the side *FB* and the angle *FBC* are common, and the side *FE* is equal to the side *FD*, but the triangles are not equal in all respects. In certain cases, however, the triangles will be equal in all respects, as will be seen from a proposition which we shall now demonstrate.

If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles opposite to a pair of equal sides equal; then if the angles opposite to the other pair of equal sides be both acute, or both obtuse, or if one of them be a right angle, the two triangles are equal in all respects.

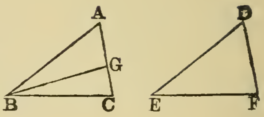


Let *ABC* and *DEF* be two triangles; let *AB* be equal to *DE*, and *BC* equal to *EF*, and the angle *A* equal to the angle *D*.

First, suppose the angles *C* and *F* acute angles.

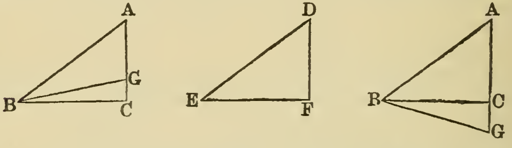
If the angle *B* be equal to the angle *E*, the triangles *ABC*, *DEF* are equal in all respects, by I. 4. If the angle *B* be not equal to the angle *E*, one of them must be greater than the other; suppose the angle *B* greater than the angle *E*, and make the angle *ABG* equal to the angle *E*. Then the triangles *ABG*, *DEF* are equal in all respects, by I. 26; therefore *BG* is equal to *EF*, and the angle *BGA* is equal to the angle *EFD*. But the angle *EFD* is acute, by hypothesis; therefore the angle *BGA* is acute. Therefore the angle *BGC* is obtuse, by I. 13. But it has

been shewn that BG is equal to EF; and EF is equal to BC, by hypothesis; therefore BG is equal to BC. Therefore the angle BGC is equal to the angle BCG, by I. 5; and the angle BCG is acute, by hypothesis; therefore the angle BGC is acute. But BGC was shewn to be obtuse; which is absurd. Therefore the angles ABC, DEF are not unequal; that is, they are equal. Therefore the triangles ABC, DEF are equal in all respects, by I. 4.



Next, suppose the angles at C and F obtuse angles. The demonstration is similar to the above.

Lastly, suppose one of the angles a right angle, namely, the angle C. If the angle B be not equal to the angle E, make the



angle ABG equal to the angle E. Then it may be shewn, as before, that BG is equal to BC, and therefore the angle BGC is equal to the angle BCG, that is, equal to a right angle. Therefore two angles of the triangle BGC are equal to two right angles; which is impossible, by I. 17. Therefore the angles ABC and DEF are not unequal; that is, they are equal. Therefore the triangles ABC, DEF are equal in all respects, by I. 4.

If the angles A and D are both right angles, or both obtuse, the angles C and F must be both acute, by I. 17. If AB is less than BC, and DE less than EF, the angles at C and F must be both acute, by I. 18 and I. 17.

The propositions from I. 27 to I. 34 inclusive may be said to constitute the second section of the first Book of the *Elements*. They relate to the theory of parallel straight lines. In I. 29 Euclid uses for the first time his twelfth axiom. The theory of parallel straight lines has always been considered the great difficulty of elementary geometry, and many attempts have been made

to overcome this difficulty in a better way than Euclid has done. We shall not give an account of these attempts. The student who wishes to examine them may consult Camerer's *Euclid*, Gergonne's *Annales de Mathématiques*, Volumes XV and XVI, the work by Colonel Perronet Thompson entitled *Geometry without Axioms*, the article *Parallels* in the *English Cyclopædia*, a memoir by Professor Baden Powell in the second volume of the *Memoirs of the Ashmolean Society*, an article by M. Bouniakofsky in the *Bulletin de l'Académie Impériale*, Volume V, St Pétersbourg, 1863, articles in the volumes of the *Philosophical Magazine* for 1856 and 1857, and a dissertation entitled *Sur un point de l'histoire de la Géométrie chez les Grecs......par A. J. H. Vincent. Paris*, 1857.

Speaking generally it may be said that the methods which differ substantially from Euclid's involve, in the first place an axiom as difficult as his, and then an intricate series of propositions; while in Euclid's method after the axiom is once admitted the remaining process is simple and clear.

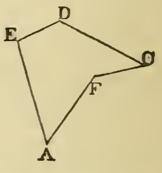
One modification of Euclid's axiom has been proposed, which appears to diminish the difficulty of the subject. This consists in assuming instead of Euclid's axiom the following; *two intersecting straight lines cannot be both parallel to a third straight line.* The propositions in the *Elements* are then demonstrated as in Euclid up to I. 28, inclusive. Then, in I. 29, we proceed with Euclid up to the words, "therefore the angles BGH, GHD are less than two right angles." We then infer that BGH and GHD must meet: because if a straight line be drawn through G so as to make the interior angles together equal to two right angles this straight line will be parallel to CD, by I. 28; and, by our axiom, there cannot be two parallels to CD, both passing through G.

This form of making the necessary assumption has been recommended by various eminent mathematicians, among whom may be mentioned Playfair and De Morgan. By postponing the consideration of the axiom until it is wanted, that is, until after I. 28, and then presenting it in the form here given, the theory of parallel straight lines appears to be treated in the easiest manner that has hitherto been proposed.

I. 30. Here we may in the same way shew that if AB and EF are each of them parallel to CD, they are parallel to each other. It has been said that the case considered in the text is so obvious as to need no demonstration; for if AB and CD can

never meet EF, which lies between them, they cannot meet one another.

I. 32. The corollaries to I. 32 were added by Simson. In the second corollary it ought to be stated what is meant by an *exterior* angle of a rectilineal figure. At each angular point let *one* of the sides meeting at that point be produced; then the exterior angle at that point is the angle contained between this produced part and the side which is not produced. *Either* of the sides may be produced, for the two angles which can thus be obtained are equal, by I. 15.



The rectilineal figures to which Euclid confines himself are those in which the angles all face inwards; we may here however notice another class of figures. In the accompanying diagram the angle AFC faces outwards, and it is an angle less than two right angles; this angle however is not one of the interior angles of the figure $AEDCF$. We may consider the corresponding interior angle to be the excess of four right angles above the angle AFC; such an angle, greater than two right angles, is called a *re-entrant* angle.

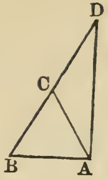
The *first* of the corollaries to I. 32 is true for a figure which has a re-entrant angle or re-entrant angles; but the *second* is not.

I. 32. If two triangles have two angles of the one equal to two angles of the other each to each they shall also have their third angles equal. This is a very important result, which is often required in the *Elements*. The student should notice how this result is established on Euclid's principles. By Axioms 11 and 2 one pair of right angles is equal to any other pair of right angles. Then, by I. 32, the three angles of one triangle are together equal to the three angles of any other triangle. Then, by Axiom 2, the sum of the two angles of one triangle is equal to the sum of the two equal angles of the other; and then, by Axiom 3, the third angles are equal.

After I. 32 we can draw a straight line at right angles to a given straight line from its extremity, without producing the given straight line.

Let AB be the given straight line. It is required to draw from A a straight line at right angles to AB.

On AB describe the equilateral triangle ABC. Produce BC to D, so that CD may be equal to CB. Join AD. Then AD shall be at right angles to AB. For, the angle CAD is equal to the angle CDA, and the angle CAB is equal to the angle CBA, by I. 5. Therefore the angle BAD is equal to the two angles ABD, BDA, by Axiom 2. Therefore the angle BAD is a right angle, by I. 32.



The propositions from I. 35 to I. 48 inclusive may be said to constitute the third section of the first Book of the *Elements*. They relate to equality of area in figures which are not necessarily identical in form.

I. 35. Here Simson has altered the demonstration given by Euclid, because, as he says, there would be three cases to consider in following Euclid's method. Simson however uses the third Axiom in a peculiar manner, when he first takes a triangle from a trapezium, and then another triangle from the *same* trapezium, and infers that the remainders are equal. If the demonstration is to be conducted strictly after Euclid's manner, three cases must be made, by dividing the latter part of the demonstration into two. In the left-hand figure we may suppose the point of intersection of BE and DC to be denoted by G. Then, the triangle ABE is equal to the triangle DCF; take away the triangle DGE from each; then the figure $ABGD$ is equal to the figure $EGCF$; add the triangle GBC to each; then the parallelogram $ABCD$ is equal to the parallelogram $EBCF$. In the right-hand figure we have the triangle AEB equal to the triangle DFC; add the figure $BEDC$ to each; then the parallelogram $ABCD$ is equal to the parallelogram $EBCF$.

The equality of the parallelograms in I. 35 is an equality of area, and not an identity of figure. Legendre proposed to use the word *equivalent* to express the equality of area, and to restrict the word *equal* to the case in which magnitudes admit of superposition and coincidence. This distinction, however, has not been generally adopted, probably because there are few cases in which any ambiguity can arise; in such cases we may say especially, *equal in area*, to prevent misconception.

Cresswell, in his *Treatise of Geometry*, has given a demonstration of I. 35 which shews that the parallelograms may be

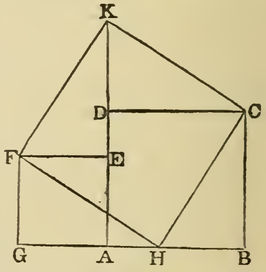
divided into pairs of pieces admitting of superposition and coincidence; see also his Preface, page x.

I. 38. An important case of I. 38 is that in which the triangles are on equal bases and have a *common* vertex.

I. 40. We may demonstrate I. 40 without adopting the indirect method. Join *BD, CD*. The triangles *DBC* and *DEF* are equal, by I. 38; the triangles *ABC* and *DEF* are equal, by hypothesis; therefore the triangles *DBC* and *ABC* are equal, by the first Axiom. Therefore *AD* is parallel to *BC*, by I. 39. *Philosophical Magazine*, October 1850.

I. 44. In I. 44, Euclid does not shew that *AH* and *FG* will meet. "I cannot help being of opinion that the construction would have been more in Euclid's manner if he had made *GH* equal to *BA* and then joining *HA* had proved that *HA* was parallel to *GB* by the thirty-third proposition." *Williamson.*

I. 47. Tradition ascribed the discovery of I. 47 to Pythagoras. Many demonstrations have been given of this celebrated proposition; the following is one of the most interesting.



Let *ABCD, AEFG* be any two squares, placed so that their bases may join and form one straight line. Take *GH* and *EK* each equal to *AB*, and join *HC, CK, KF, FH*.

Then it may be shewn that the triangle *HBC* is equal in all respects to the triangle *FEK*, and the triangle *KDC* to the triangle *FGH*. Therefore the two squares are together equivalent to the figure *CKFH*. It may then be shewn, with the aid of I. 32, that the figure *CKFH* is a square. And the side *CH* is the hypotenuse of a right-angled triangle of which the sides *CB, BH* are equal to the sides of the two given squares. This demonstration requires no proposition of Euclid after I. 32, and it shews how two given squares may be cut into pieces which will fit together so as to form a third square. *Quarterly Journal of Mathematics*, Vol. I.

A large number of demonstrations of this proposition are collected in a dissertation by Joh. Jos. Ign. Hoffmann, entitled *Der Pythagorische Lehrsatz....Zweyte...Ausgabe. Mainz.* 1821.

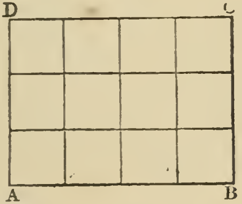
THE SECOND BOOK.

THE second book is devoted to the investigation of relations between the rectangles contained by straight lines divided into segments in various ways.

When a straight line is divided into two parts, each part is called a segment by Euclid. It is found convenient to extend the meaning of the word *segment*, and to lay down the following definition. When a point is taken in a straight line, or in the straight line produced, the distances of the point from the ends of the straight line are called segments of the straight line. When it is necessary to distinguish them, such segments are called *internal* or *external*, according as the point is in the straight line, or in the straight line produced.

The student cannot fail to notice that there is an analogy between the first ten propositions of this book and some elementary facts in Arithmetic and Algebra.

Let $ABCD$ represent a rectangle which is 4 inches long and 3 inches broad. Then, by drawing straight lines parallel to the sides, the figure may be divided into 12 squares, each square being described on a side which represents an inch in length. A square described on a side measuring an inch is called, for shortness, a *square inch*. Thus if a rectangle is 4 inches long and 3 inches broad it may be divided into 12 square inches; this is expressed by saying, that its area is equal to 12 square inches, or, more briefly, that it contains 12 square inches. And a similar result is easily seen to hold in all similar cases. Suppose, for example, that a rectangle is 12 feet long and 7 feet broad; then its area is equal to 12 times 7 square feet, that is to 84 square feet; this may be expressed briefly in common language thus; if a rectangle measures 12 feet by 7 it contains 84 square feet. It must be carefully observed that the sides of the rectangle are supposed to be measured by the same unit of length. Thus if a rectangle is a yard in length, and a foot and a half in breadth, we



must express each of these dimensions in terms of the same unit; we may say that the rectangle measures 36 inches by 18 inches, and contains 36 times 18 square inches, that is, 648 square inches.

Thus universally, if one side of a rectangle contain a unit of length an exact number of times, and if an adjacent side of the rectangle also contain the same unit of length an exact number of times, the product of these numbers will be the number of square units contained in the rectangle.

Next suppose we have a *square*, and let its side be 5 inches in length. Then, by our rule, the area of the square is 5 times 5 square inches, that is 25 square inches. Now the number 25 is called in Arithmetic the square of the number 5. And universally, if a straight line contain a unit of length an exact number of times, the area of the square described *on* the straight line is denoted by the square *of* the number which denotes the length of the straight line.

Thus we see that there is in general a connexion between the product of two numbers and the rectangle contained by two straight lines, and in particular a connexion between the square *of* a number and the square *on* a straight line; and in consequence of this connexion the first ten propositions in Euclid's Second Book correspond to propositions in Arithmetic and Algebra.

The student will perceive that we speak of the square described *on* a straight line, when we refer to the geometrical figure, and of the square *of* a number when we refer to Arithmetic. The editors of Euclid generally use the words "square described *upon*" in I. 47 and I. 48, and afterwards speak of the square *of* a straight line. Euclid himself retains throughout the same form of expression, and we have imitated him.

Some editors of Euclid have added Arithmetical or Algebraical demonstrations of the propositions in the second book, founded on the connexion we have explained. We have thought it unnecessary to do this, because the student who is acquainted with the elements of Arithmetic and Algebra will find no difficulty in supplying such demonstrations himself, so far as they are usually given. We say *so far as they are usually given*, because these demonstrations usually imply that the sides of rectangles can always be expressed *exactly* in terms of some unit of length; whereas the student will find hereafter that this is not the case, owing to the existence of what are technically called *incommensurable* magnitudes. We do not enter on this subject

as it would lead us too far from Euclid's *Elements of Geometry*, with which we are here occupied.

The first ten propositions in the second book of Euclid may be arranged and enunciated in various ways; we will briefly indicate this, but we do not consider it of any importance to distract the attention of a beginner with these diversities.

II. 2 and II. 3 are particular cases of II. 1.

II. 4 is very important; the following particular case of it should be noticed; *the square described on a straight line made up of two equal straight lines is equal to four times the square described on one of the two equal straight lines.*

II. 5 and II. 6 may be included in one enunciation thus; *the rectangle under the sum and difference of two straight lines is equal to the difference of the squares described on those straight lines;* or thus, *the rectangle contained by two straight lines together with the square described on half their difference, is equal to the square described on half their sum.*

II. 7 may be enunciated thus; *the square described on a straight line which is the difference of two other straight lines is less than the sum of the squares described on those straight lines by twice the rectangle contained by those straight lines.* Then from this and II. 4, and the second Axiom, we infer that *the square described on the sum of two straight lines, and the square described on their difference, are together double of the sum of the squares described on the straight lines;* and this enunciation includes both II. 9 and II. 10, so that the demonstrations given of these propositions by Euclid might be superseded.

II. 8 coincides with the second form of enunciation which we have given to II. 5 and II. 6, bearing in mind the particular case of II. 4 which we have noticed.

II. 11. When the student is acquainted with the elements of Algebra he should notice that II. 11 gives a geometrical construction for the solution of a particular quadratic equation.

II. 12, II. 13. These are interesting in connexion with I. 47; and, as the student may see hereafter, they are of great importance in Trigonometry; they are however not required in any of the parts of Euclid's *Elements* which are usually read. The converse of I. 47 is proved in I. 48; and we can easily shew that converses of II. 12 and II. 13 are true.

Take the following, which is the converse of II. 12; *if the square described on one side of a triangle be greater than the sum*

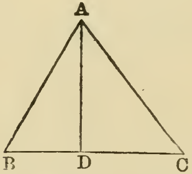
of the squares described on the other two sides, the angle opposite to the first side is obtuse.

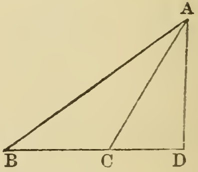
For the angle cannot be a right angle, since the square described on the first side would then be equal to the sum of the squares described on the other two sides, by I. 47; and the angle cannot be acute, since the square described on the first side would then be less than the sum of the squares described on the other two sides, by II. 13; therefore the angle must be obtuse.

Similarly we may demonstrate the following, which is the converse of II. 13; *if the square described on one side of a triangle be less than the sum of the squares described on the other two sides, the angle opposite to the first side is acute.*

II. 13. Euclid enunciates II. 13 thus; *in acute-angled triangles*, &c.; and he gives only the first case in the demonstration. But, as Simson observes, the proposition holds for any triangle; and accordingly Simson supplies the second and third cases. It has, however, been often noticed that the same demonstration is applicable to the first and second cases; and it would be a great improvement as to brevity and clearness to take these two cases together. Then the whole demonstration will be as follows.

Let ABC be any triangle, and the angle at B one of its acute angles; and, if AC be not perpendicular to BC, let fall on BC, produced if necessary, the perpendicular AD from the opposite angle: the square on AC opposite to the angle B, shall be less than the squares on CB, BA, by twice the rectangle CB, BD.





First, suppose AC not perpendicular to BC.

The squares on CB, BD are equal to twice the rectangle CB, BD, together with the square on CD. [II. 7.

To each of these equals add the square on DA.

Therefore the squares on CB, BD, DA are equal to twice the rectangle CB, BD, together with the squares on CD, DA.

But the square on AB is equal to the squares on BD, DA,

and the square on AC is equal to the squares on CD, DA, because the angle BDA is a right angle. [I. 47.
Therefore the squares on CB, BA are equal to the square on AC, together with twice the rectangle CB, BD;
that is, the square on AC alone is less than the squares on CB, BA, by twice the rectangle CB, BD.

Next, suppose AC perpendicular to BC. Then BC is the straight line intercepted between the perpendicular and the acute angle at B.

And the square on AB is equal to the squares on AC, CB. [I. 47.
Therefore the square on AC is less than the squares on AB, BC, by twice the square on BC.

II. 14. This is not required in any of the parts of Euclid's *Elements* which are usually read; it is included in VI. 22.

EXERCISES IN EUCLID.

I. 1 to 15.

1. On a given straight line describe an isosceles triangle having each of the sides equal to a given straight line.

2. In the figure of I. 2 if the diameter of the smaller circle is the radius of the larger, shew where the given point and the vertex of the constructed triangle will be situated.

3. If two straight lines bisect each other at right angles, any point in either of them is equidistant from the extremities of the other.

4. If the angles ABC and ACB at the base of an isosceles triangle be bisected by the straight lines BD, CD, shew that DBC will be an isosceles triangle.

5. BAC is a triangle having the angle B double of the angle A. If BD bisects the angle B and meets AC at D, shew that BD is equal to AD.

6. In the figure of I. 5 if FC and BG meet at H shew that FH and GH are equal.

7. In the figure of I. 5 if FC and BG meet at H, shew that AH bisects the angle BAC.

8. The sides AB, AD of a quadrilateral $ABCD$ are equal, and the diagonal AC bisects the angle BAD: shew that the sides CB and CD are equal, and that the diagonal AC bisects the angle BCD.

9. ACB, ADB are two triangles on the same side of AB, such that AC is equal to BD, and AD is equal to BC, and AD and BC intersect at O: shew that the triangle AOB is isosceles.

10. The opposite angles of a rhombus are equal.

11. A diagonal of a rhombus bisects each of the angles through which it passes.

12. If two isosceles triangles are on the same base the straight line joining their vertices, or that straight line produced, will bisect the base at right angles.

*13. Find a point in a given straight line such that its distances from two given points may be equal.

14. Through two given points on opposite sides of a given straight line draw two straight lines which shall meet in that given straight line, and include an angle bisected by that given straight line.

15. A given angle BAC is bisected; if CA is produced to G and the angle BAG bisected, the two bisecting lines are at right angles.

16. If four straight lines meet at a point so that the opposite angles are equal, these straight lines are two and two in the same straight line.

I. 16 to 26.

17. ABC is a triangle and the angle A is bisected by a straight line which meets BC at D; shew that BA is greater than BD, and CA greater than CD.

18. In the figure of I. 17 shew that ABC and ACB are together less than two right angles, by joining A to any point in BC.

19. $ABCD$ is a quadrilateral of which AD is the longest side and BC the shortest; shew that the angle ABC is greater than the angle ADC, and the angle BCD greater than the angle BAD.

20. If a straight line be drawn through A one of the angular points of a square, cutting one of the opposite sides, and meeting the other produced at F, shew that AF is greater than the diagonal of the square.

21. The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and of others, that which is nearer to the perpendicular is less than the more remote; and two, and only two, equal straight lines can be drawn from the given point to the given straight line, one on each side of the perpendicular.

22. The sum of the distances of any point from the three angles of a triangle is greater than half the sum of the sides of the triangle.

23. The four sides of any quadrilateral are together greater than the two diagonals together.

24. The two sides of a triangle are together greater than twice the straight line drawn from the vertex to the middle point of the base.

25. If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles.

26. If the angle C of a triangle is equal to the sum of the angles A and B, the side AB is equal to twice the straight line joining C to the middle point of AB.

27. Construct a triangle, having given the base, one of the angles at the base, and the sum of the sides.

28. The perpendiculars let fall on two sides of a triangle from any point in the straight line bisecting the angle between them are equal to each other.

29. In a given straight line find a point such that the perpendiculars drawn from it to two given straight lines shall be equal.

30. Through a given point draw a straight line such that the perpendiculars on it from two given points may be on opposite sides of it and equal to each other.

31. A straight line bisects the angle A of a triangle ABC; from B a perpendicular is drawn to this bisecting straight line, meeting it at D, and BD is produced to meet AC or AC produced at E: shew that BD is equal to DE.

32. AB, AC are any two straight lines meeting at A: through any point P draw a straight line meeting them at E and F, such that AE may be equal to AF.

33. Two right-angled triangles have their hypotenuses equal, and a side of one equal to a side of the other: shew that they are equal in all respects.

I. 27 to 31.

34. Any straight line parallel to the base of an isosceles triangle makes equal angles with the sides.

35. If two straight lines A and B are respectively parallel to two others C and D, shew that the inclination of A to B is equal to that of C to D.

36. A straight line is drawn terminated by two parallel straight lines; through its middle point any straight line is

drawn and terminated by the parallel straight lines. Shew that the second straight line is bisected at the middle point of the first.

37. If through any point equidistant from two parallel straight lines, two straight lines be drawn cutting the parallel straight lines, they will intercept equal portions of these parallel straight lines.

38. If the straight line bisecting the exterior angle of a triangle be parallel to the base, shew that the triangle is isosceles.

39. Find a point B in a given straight line CD, such that if AB be drawn to B from a given point A, the angle ABC will be equal to a given angle.

40. If a straight line be drawn bisecting one of the angles of a triangle to meet the opposite side, the straight lines drawn from the point of section parallel to the other sides, and terminated by these sides, will be equal.

41. The side BC of a triangle ABC is produced to a point D; the angle ACB is bisected by the straight line CE which meets AB at E. A straight line is drawn through E parallel to BC, meeting AC at F, and the straight line bisecting the exterior angle ACD at G. Shew that EF is equal to FG.

42. AB is the hypotenuse of a right-angled triangle ABC: find a point D in AB such that DB may be equal to the perpendicular from D on AC.

43. ABC is an isosceles triangle: find points D, E in the equal sides AB, AC such that BD, DE, EC may all be equal.

44. A straight line drawn at right angles to BC the base of an isosceles triangle ABC cuts the side AB at D and CA produced at E: shew that AED is an isosceles triangle.

I. 32.

45. From the extremities of the base of an isosceles triangle straight lines are drawn perpendicular to the sides; shew that the angles made by them with the base are each equal to half the vertical angle.

46. On the sides of any triangle ABC equilateral triangles BCD, CAE, ABF are described, all external: shew that the straight lines AD, BE, CF are all equal.

47. What is the magnitude of an angle of a regular octagon?

48. Through two given points draw two straight lines forming with a straight line given in position an equilateral triangle.

49. If the straight lines bisecting the angles at the base of an isosceles triangle be produced to meet, they will contain an angle equal to an exterior angle of the triangle.

50. A is the vertex of an isosceles triangle ABC, and BA is produced to D, so that AD is equal to BA; and DC is drawn: shew that BCD is a right angle.

51. ABC is a triangle, and the exterior angles at B and C are bisected by the straight lines BD, CD respectively, meeting at D: shew that the angle BDC together with half the angle BAC make up a right angle.

52. Shew that any angle of a triangle is obtuse, right, or acute, according as it is greater than, equal to, or less than the other two angles of the triangle taken together.

53. Construct an isosceles triangle having the vertical angle four times each of the angles at the base.

54. In the triangle ABC the side BC is bisected at E and AB at G; AE is produced to F so that EF is equal to AE, and CG is produced to H so that GH is equal to CG: shew that FB and HB are in one straight line.

55. Construct an isosceles triangle which shall have one-third of each angle at the base equal to half the vertical angle.

56. AB, AC are two straight lines given in position: it is required to find in them two points P and Q, such that, PQ being joined, AP and PQ may together be equal to a given straight line, and may contain an angle equal to a given angle.

57. Straight lines are drawn through the extremities of the base of an isosceles triangle, making angles with it on the side remote from the vertex, each equal to one-third of one of the equal angles of the triangle and meeting the sides produced: shew that three of the triangles thus formed are isosceles.

58. AEB, CED are two straight lines intersecting at E; straight lines AC, DB are drawn forming two triangles ACE, BED; the angles ACE, DBE are bisected by the straight lines CF, BF, meeting at F. Shew that the angle CFB is equal to half the sum of the angles EAC, EDB.

59. The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.

60. From the angle A of a triangle ABC a perpendicular is drawn to the opposite side, meeting it, produced if necessary, at D; from the angle B a perpendicular is drawn to the opposite side, meeting it, produced if necessary, at E: shew that the straight lines which join D and E to the middle point of AB are equal.

61. From the angles at the base of a triangle perpendiculars are drawn to the opposite sides, produced if necessary: shew that the straight line joining the points of intersection will be bisected by a perpendicular drawn to it from the middle point of the base.

62. In the figure of I. 1, if C and H be the points of intersection of the circles, and AB be produced to meet one of the circles at K, shew that CHK is an equilateral triangle.

63. The straight lines bisecting the angles at the base of an isosceles triangle meet the sides at D and E: shew that DE is parallel to the base.

64. AB, AC are two given straight lines, and P is a given point in the former: it is required to draw through P a straight line to meet AC at Q, so that the angle APQ may be three times the angle AQP.

65. Construct a right-angled triangle, having given the hypotenuse and the sum of the sides.

66. Construct a right-angled triangle, having given the hypotenuse and the difference of the sides.

67. Construct a right-angled triangle, having given the hypotenuse and the perpendicular from the right angle on it.

68. Construct a right-angled triangle, having given the perimeter and an angle.

69. Trisect a right angle.

70. Trisect a given finite straight line.

71. From a given point it is required to draw to two parallel straight lines, two equal straight lines at right angles to each other.

72. Describe a triangle of given perimeter, having its angles equal to those of a given triangle.

I. 33, 34.

73. If a quadrilateral have two of its opposite sides parallel, and the two others equal but not parallel, any two of its opposite angles are together equal to two right angles.

74. If a straight line which joins the extremities of two equal straight lines, not parallel, make the angles on the same side of it equal to each other, the straight line which joins the other extremities will be parallel to the first.

75. No two straight lines drawn from the extremities of the base of a triangle to the opposite sides can possibly bisect each other.

76. If the opposite sides of a quadrilateral are equal it is a parallelogram.

77. If the opposite angles of a quadrilateral are equal it is a parallelogram.

78. The diagonals of a parallelogram bisect each other.

79. If the diagonals of a quadrilateral bisect each other it is a parallelogram.

80. If the straight line joining two opposite angles of a parallelogram bisect the angles the four sides of the parallelogram are equal.

81. Draw a straight line through a given point such that the part of it intercepted between two given parallel straight lines may be of given length.

82. Straight lines bisecting two adjacent angles of a parallelogram intersect at right angles.

83. Straight lines bisecting two opposite angles of a parallelogram are either parallel or coincident.

84. If the diagonals of a parallelogram are equal all its angles are equal.

85. Find a point such that the perpendiculars let fall from it on two given straight lines shall be respectively equal to two given straight lines. How many such points are there?

86. It is required to draw a straight line which shall be equal to one straight line and parallel to another, and be terminated by two given straight lines.

87. On the sides AB, BC, and CD of a parallelogram $ABCD$ three equilateral triangles are described, that on BC towards the same parts as the parallelogram, and those on AB, CD towards the opposite parts: shew that the

distances of the vertices of the triangles on AB, CD from that on BC are respectively equal to the two diagonals of the parallelogram.

88. If the angle between two adjacent sides of a parallelogram be increased, while their lengths do not alter, the diagonal through their point of intersection will diminish.

89. A, B, C are three points in a straight line, such that AB is equal to BC: shew that the sum of the perpendiculars from A and C on any straight line which does not pass between A and C is double the perpendicular from B on the same straight line.

90. If straight lines be drawn from the angles of any parallelogram perpendicular to any straight line which is outside the parallelogram, the sum of those from one pair of opposite angles is equal to the sum of those from the other pair of opposite angles.

91. If a six-sided plane rectilineal figure have its opposite sides equal and parallel, the three straight lines joining the opposite angles will meet at a point.

92. AB, AC are two given straight lines; through a given point E between them it is required to draw a straight line GEH such that the intercepted portion GH shall be bisected at the point E.

93. Inscribe a rhombus within a given parallelogram, so that one of the angular points of the rhombus may be at a given point in a side of the parallelogram.

94. $ABCD$ is a parallelogram, and E, F, the middle points of AD and BC respectively; shew that BE and DF will trisect the diagonal AC.

I. 35 to 45.

95. $ABCD$ is a quadrilateral having BC parallel to AD; shew that its area is the same as that of the parallelogram which can be formed by drawing through the middle point of DC a straight line parallel to AB.

96. $ABCD$ is a quadrilateral having BC parallel to AD, E is the middle point of DC; shew that the triangle AEB is half the quadrilateral.

97. Shew that any straight line passing through the middle point of the diameter of a parallelogram and terminated by two opposite sides, bisects the parallelogram.

98. Bisect a parallelogram by a straight line drawn through a given point within it.

99. Construct a rhombus equal to a given parallelogram.

100. If two triangles have two sides of the one equal to two sides of the other, each to each, and the sum of the two angles contained by these sides equal to two right angles, the triangles are equal in area.

101. A straight line is drawn bisecting a parallelogram $ABCD$ and meeting AD at E and BC at F: shew that the triangles EBF and CED are equal.

102. Shew that the four triangles into which a parallelogram is divided by its diagonals are equal in area.

103. Two straight lines AB and CD intersect at E, and the triangle AEC is equal to the triangle BED: shew that BC is parallel to AD.

104. $ABCD$ is a parallelogram; from any point P in the diagonal BD the straight lines PA, PC are drawn. Shew that the triangles PAB and PCB are equal.

105. If a triangle is described having two of its sides equal to the diagonals of any quadrilateral, and the included angle equal to either of the angles between these diagonals, then the area of the triangle is equal to the area of the quadrilateral.

106. The straight line which joins the middle points of two sides of any triangle is parallel to the base.

107. Straight lines joining the middle points of adjacent sides of a quadrilateral form a parallelogram.

108. D, E are the middle points of the sides AB, AC of a triangle, and CD, BE intersect at F: shew that the triangle BFC is equal to the quadrilateral $ADFE$.

109. The straight line which bisects two sides of any triangle is half the base.

110. In the base AC of a triangle take any point D; bisect AD, DC, AB, BC at the points E, F, G, H respectively: shew that EG is equal and parallel to FH.

111. Given the middle points of the sides of a triangle, construct the triangle.

112. If the middle points of any two sides of a triangle be joined, the triangle so cut off is one quarter of the whole.

113. The sides AB, AC of a given triangle ABC are bisected at the points E, F; a perpendicular is drawn from A to the opposite side, meeting it at D. Shew that the

angle FDE is equal to the angle BAC. Shew also that $AFDE$ is half the triangle ABC.

114. Two triangles of equal area stand on the same base and on opposite sides: shew that the straight line joining their vertices is bisected by the base or the base produced.

115 Three parallelograms which are equal in all respects are placed with their equal bases in the same straight line and contiguous; the extremities of the base of the first are joined with the extremities of the side opposite to the base of the third, towards the same parts: shew that the portion of the new parallelogram cut off by the second is one half the area of any one of them.

116. $ABCD$ is a parallelogram; from D draw any straight line DFG meeting BC at F and AB produced at G; draw AF and CG: shew that the triangles ABF, CFG are equal.

117. ABC is a given triangle: construct a triangle of equal area, having for its base a given straight line AD, coinciding in position with AB.

118. ABC is a given triangle: construct a triangle of equal area, having its vertex at a given point in BC and its base in the same straight line as AB.

119. $ABCD$ is a given quadrilateral: construct another quadrilateral of equal area having AB for one side, and for another a straight line drawn through a given point in CD parallel to AB.

120. $ABCD$ is a quadrilateral: construct a triangle whose base shall be in the same straight line as AB, vertex at a given point P in CD, and area equal to that of the given quadrilateral.

121. ABC is a given triangle: construct a triangle of equal area, having its base in the same straight line as AB, and its vertex in a given straight line parallel to AB.

122. Bisect a given triangle by a straight line drawn through a given point in a side.

123. Bisect a given quadrilateral by a straight line drawn through a given angular point.

124. If through the point O within a parallelogram $ABCD$ two straight lines are drawn parallel to the sides, and the parallelograms OB and OD are equal, the point O is in the diagonal AC.

I. 46 to 48.

125. On the sides AC, BC of a triangle ABC, squares $ACDE$, $BCFH$ are described: shew that the straight lines AF and BD are equal.

126. The square on the side subtending an acute angle of a triangle is less than the squares on the sides containing the acute angle.

127. The square on the side subtending an obtuse angle of a triangle is greater than the squares on the sides containing the obtuse angle.

128. If the square on one side of a triangle be less than the squares on the other two sides, the angle contained by these sides is an acute angle; if greater, an obtuse angle.

129. A straight line is drawn parallel to the hypotenuse of a right-angled triangle, and each of the acute angles is joined with the points where this straight line intersects the sides respectively opposite to them: shew that the squares on the joining straight lines are together equal to the square on the hypotenuse and the square on the straight line drawn parallel to it.

130. If any point P be joined to A, B, C, D, the angular points of a rectangle, the squares on PA and PC are together equal to the squares on PB and PD.

131. In a right-angled triangle if the square on one of the sides containing the right angle be three times the square on the other, and from the right angle two straight lines be drawn, one to bisect the opposite side, and the other perpendicular to that side, these straight lines divide the right angle into three equal parts.

132. If ABC be a triangle whose angle A is a right angle, and BE, CF be drawn bisecting the opposite sides respectively, shew that four times the sum of the squares on BE and CF is equal to five times the square on BC.

133. On the hypotenuse BC, and the sides CA, AB of a right-angled triangle ABC, squares $BDEC$, AF, and AG are described: shew that the squares on DG and EF are together equal to five times the square on BC.

II. 1 to 11.

134. A straight line is divided into two parts; shew that if twice the rectangle of the parts is equal to the sum of the squares described on the parts, the straight line is bisected.

135. Divide a given straight line into two parts such that the rectangle contained by them shall be the greatest possible.

136. Construct a rectangle equal to the difference of two given squares.

137. Divide a given straight line into two parts such that the sum of the squares on the two parts may be the least possible.

138. Shew that the square on the sum of two straight lines together with the square on their difference is double the squares on the two straight lines.

139. Divide a given straight line into two parts such that the sum of their squares shall be equal to a given square.

140. Divide a given straight line into two parts such that the square on one of them may be double the square on the other.

141. In the figure of II. 11 if CH be produced to meet BF at L, shew that CL is at right angles to BF.

142. In the figure of II. 11 if BE and CH meet at O, shew that AO is at right angles to CH.

143. Shew that in a straight line divided as in II. 11 the rectangle contained by the sum and difference of the parts is equal to the rectangle contained by the parts.

II. 12 to 14.

144. The square on the base of an isosceles triangle is equal to twice the rectangle contained by either side and by the straight line intercepted between the perpendicular let fall on it from the opposite angle and the extremity of the base.

145. In any triangle the sum of the squares on the sides is equal to twice the square on half the base together with twice the square on the straight line drawn from the vertex to the middle point of the base.

146. ABC is a triangle having the sides AB and AC equal; if AB is produced beyond the base to D so that BD is equal to AB, shew that the square on CD is equal to the square on AB, together with twice the square on BC.

147. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

148. The base of a triangle is given and is bisected by the centre of a given circle: if the vertex be at any point of the circumference, shew that the sum of the squares on the two sides of the triangle is invariable.

149. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides.

150. If a circle be described round the point of intersection of the diameters of a parallelogram as a centre, shew that the sum of the squares on the straight lines drawn from any point in its circumference to the four angular points of the parallelogram is constant.

151. The squares on the sides of a quadrilateral are together greater than the squares on its diagonals by four times the square on the straight line joining the middle points of its diagonals.

152. In AB the diameter of a circle take two points C and D equally distant from the centre, and from any point E in the circumference draw EC, ED: shew that the squares on EC and ED are together equal to the squares on AC and AD.

153. In BC the base of a triangle take D such that the squares on AB and BD are together equal to the squares on AC and CD, then the middle point of AD will be equally distant from B and C.

154. The square on any straight line drawn from the vertex of an isosceles triangle to the base is less than the square on a side of the triangle by the rectangle contained by the segments of the base.

155. A square $BDEC$ is described on the hypotenuse BC of a right-angled triangle ABC: shew that the squares on DA and AC are together equal to the squares on EA and AB.

156. ABC is a triangle in which C is a right angle, and DE is drawn from a point D in AC perpendicular to

AB: shew that the rectangle *AB*, *AE* is equal to the rectangle *AC*, *AD*.

157. If a straight line be drawn through one of the angles of an equilateral triangle to meet the opposite side produced, so that the rectangle contained by the whole straight line thus produced and the part of it produced is equal to the square on the side of the triangle, shew that the square on the straight line so drawn will be double the square on a side of the triangle.

158. In a triangle whose vertical angle is a right angle a straight line is drawn from the vertex perpendicular to the base: shew that the square on this perpendicular is equal to the rectangle contained by the segments of the base.

159. In a triangle whose vertical angle is a right angle a straight line is drawn from the vertex perpendicular to the base: shew that the square on either of the sides adjacent to the right angle is equal to the rectangle contained by the base and the segment of it adjacent to that side.

160. In a triangle *ABC* the angles *B* and *C* are acute: if *E* and *F* be the points where perpendiculars from the opposite angles meet the sides *AC*, *AB*, shew that the square on *BC* is equal to the rectangle *AB*, *BF*, together with the rectangle *AC*, *CE*.

161. Divide a given straight line into two parts so that the rectangle contained by them may be equal to the square described on a given straight line which is less than half the straight line to be divided.

III. 1 to 15.

162. Describe a circle with a given centre cutting a given circle at the extremities of a diameter.

163. Shew that the straight lines drawn at right angles to the sides of a quadrilateral inscribed in a circle from their middle points intersect at a fixed point.

164. If two circles cut each other, any two parallel straight lines drawn through the points of section to cut the circles are equal.

165. Two circles whose centres are *A* and *B* intersect at *C*; through *C* two chords *DCE* and *FCG* are drawn equally inclined to *AB* and terminated by the circles: shew that *DF* and *FG* are equal.